**Prof. A. Suciu** · **MTH U371–LINEAR ALGEBRA** · **Spring 2005**

# Sample Quiz 4, with Solutions

**1.** Let $A$ be an orthogonal $n \times n$ matrix (recall this means that the columns of $A$ are orthonormal), and let $A^\top$ be its transpose.

(a) Find:

$AA^\top = I_n$ [This is equivalent to columns of $A$ being orthonormal].

$A^\top A = I_n$ [Since, by the above, $A^\top = A^{-1}$, and we know that $A^{-1}A = I_n$].

(b) Find:

$\dim(\ker A) = 0$ [Since $A$ is invertible, and so $\ker A = \{\vec{0}\}$].

$\dim(\operatorname{im} A) = n$ [Since $\dim(\operatorname{im} A) + \dim(\ker A) = n$].

(c) Is $A^\top$ also orthogonal? Explain your answer.

Yes, since $A$ orthogonal $\longleftrightarrow AA^\top = I_n \longleftrightarrow A^\top (A^\top)^\top = A^\top A = I_n \longleftrightarrow A^\top$ orthogonal.

(d) Are the rows of $A$ orthonormal? Explain your answer.

Yes, since the rows of $A$ are the columns of $A^\top$, which we just saw is orthogonal.

(e) The $QR$-factorization of $A$ is given by:

$Q = A$ [Since $A$ is already orthogonal, so its columns make up $Q$].

$R = I_n$ [Since $A = QR$ gives $R = Q^{-1}A = A^{-1}A = I_n$].

**2.** Find all orthogonal matrices of the form

$$A = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & a \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & b \\ \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & c \end{bmatrix}$$

The orthonormality conditions $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$, $\vec{v}_3 \cdot \vec{v}_3 = 1$ yield:

$$\begin{aligned} a - \tfrac{1}{2}b + \tfrac{1}{2}c &= 0 \\ a + b - c &= 0 \\ a^2 + b^2 + c^2 &= 1 \end{aligned}$$

The first two equations give $b = c$ and $a = c - b = 0$. Plugging into the third equation gives $2b^2 = 1$, and so $b = \pm\frac{1}{\sqrt{2}}$. The two solutions for $A$ are:

$$A = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

**3.** Let $\vec{v}_1 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$

(a) Find the lengths of $\vec{v}_1$ and $\vec{v}_2$, and compute the dot product $\vec{v}_1 \cdot \vec{v}_2$.

$$\begin{aligned} ||\vec{v}_1|| &= \sqrt{5^2 + 12^2} = 13 \\ ||\vec{v}_2|| &= \sqrt{4^2 + (-3)^2} = 5 \\ \vec{v}_1 \cdot \vec{v}_2 &= 5 \cdot 4 + 12 \cdot (-3) = -16 \end{aligned}$$

(b) Find unit vectors in the direction of $\vec{v}_1$ and $\vec{v}_2$, respectively.

$$\begin{aligned} \vec{w}_1 &= \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{bmatrix} \frac{5}{13} \\ \frac{12}{13} \end{bmatrix} \\ \vec{w}_2 &= \frac{\vec{v}_2}{||\vec{v}_2||} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} \end{aligned}$$

(c) Find the angle between $\vec{v}_1$ and $\vec{v}_2$.

We have: $\cos\theta = \dfrac{\vec{v}_1 \cdot \vec{v}_2}{||\vec{v}_1|| \cdot ||\vec{v}_2||} = -\dfrac{16}{65}$, and so $\theta = 1.81951$ radians (or, $\theta = 104.25°$).

(d) Find the projection of $\vec{v}_2$ onto the subspace of $\mathbb{R}^2$ spanned by $\vec{v}_1$.

$\operatorname{proj}_{V_1} \vec{v}_2 = (\vec{w}_1 \cdot \vec{v}_2) \cdot \vec{w}_1 = -\dfrac{16}{13} \cdot \dfrac{1}{13} \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{80}{169} \\ -\frac{192}{169} \end{bmatrix}.$

(e) Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. Use the Gram-Schmidt process to find the $QR$-factorization of $A$.

We have:

$$\begin{aligned} \vec{a}_2 &= \vec{v}_2 - \operatorname{proj}_{V_1} \vec{v}_2 = \begin{bmatrix} \frac{756}{169} \\ -\frac{315}{169} \end{bmatrix} \\ ||\vec{a}|| &= \frac{63}{13} \\ \vec{w}_2 &= \frac{\vec{a}_2}{||\vec{a}_2||} = \begin{bmatrix} \frac{12}{13} \\ -\frac{5}{13} \end{bmatrix} \end{aligned}$$

Thus:

$$\begin{aligned} Q &= \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix} && = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix} \\ R &= \begin{bmatrix} ||\vec{v}_1|| & \vec{w}_1 \cdot \vec{v}_2 \\ 0 & ||\vec{v}_2|| \end{bmatrix} && = \begin{bmatrix} 13 & -\frac{16}{13} \\ 0 & \frac{63}{13} \end{bmatrix} \end{aligned}$$