PRACTICE QUIZ 4

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1. Consider the independent vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find an orthonormal basis $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ for the subspace of $\mathbb{R}^4$ which has $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ as a basis.

2. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$

(a) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ which has $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ as a basis.

(b) Use part (a) to find the QR-factorization of the matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$.

3. Find the QR-factorization of the matrix $A = \begin{bmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{bmatrix}$.

4. Apply the Gram-Schmidt process to the vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and write the result in the form $A = Q \cdot R$.

5. Find the matrix of the orthogonal projection onto the line in $\mathbb{R}^5$ spanned by the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} -3 & 4 \\ 9 & -12 \end{bmatrix}$.

(a) Find a basis for $\ker A$.

(b) Find a basis for $(\ker A)^\perp$.

(c) Find a basis for $\ker A^\top$.

(d) Find a basis for $(\ker A^\top)^\perp$.

7. Find all pairs of orthonormal vectors of the form $\vec{v}_1 = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ 0 \\ c \end{bmatrix}$

8. Let A be an $n \times n$ matrix. Is the matrix $AA^\top$ symmetric? Justify your answer.

9. Let A be an $n \times n$ matrix. Is the matrix $A - A^\top$ symmetric? Justify your answer.

10. Let A be an $n \times n$ matrix, and let B be a symmetric matrix. Is the matrix $A^\top BA$ symmetric? Justify your answer.

11. Which of the following statements are true, for **all** $n \times n$ orthogonal matrices A?

(a) $\operatorname{rref} A = I_n$

(b) $\ker A = \{\vec{0}\}$

(c) $\operatorname{im} A = \{\vec{0}\}$

(d) $A \cdot A^\top = A^\top \cdot A$

(e) $A^{-1} = A$

(f) $(A^\top)^{-1} = A$

12. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$.

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A \cdot \vec{x} = \vec{b}$.

(b) Find the 3×3 matrix associated with the projection of $\mathbb{R}^3$ onto the subspace $\operatorname{im} A$.

(c) Find the projection of the vector $\vec{b}$ onto $\operatorname{im} A$.

13. Find the least squares solution of the inconsistent system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Determine the error $||\vec{b} - A\vec{x}||$.

14. Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Determine the error $||\vec{b} - A\vec{x}||$.

15. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the least squares solution $\left[\begin{smallmatrix} \overline{x} \\ \overline{y} \end{smallmatrix}\right]$ of the inconsistent system $A \cdot \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \mathbf{b}$.

(b) Use your answer to part (b) to find the projection of $\mathbf{b}$ onto $\operatorname{im} A$.

16. A company gathers the following data:

Year	1993	1994	1995	1996	1997
Annual Sales (in millions of dollars)	0.8	2	3	4.2	5

Represent the years $1993, \ldots, 1997$ as $-2, -1, 0, 1, 2$, respectively, and let x denote the year. Let y denote the annual sales (in millions of dollars).

(a) Find the least squares line relating x and y.

(b) Use the equation obtained in part (a) to estimate the annual sales for the year 1999.

17. Find the equation of the least-squares line that fits the following data points:

x	1	2	4	5
y	0	1	2	3

Sketch the resulting line. What is the predicted value of y at $x = 6$, based on this model?