1	2	3	4	5	6	7	8	Σ

Name: ____________________

NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH U371 FINAL EXAM **Spring 2005**

Instructions: Put your name in the blanks above. Put your final answers to each question in the designated spaces. Calculators are permitted. A single sheet of formulas is allowed. **Show your work.** If there is not enough room to show your work, use the back page.

---

1. 12 points Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -6 \\ 8 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Are the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ linearly independent or dependent? If they are independent, explain why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} 10 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

2. 10 points The matrix $A = \begin{bmatrix} 1 & -3 & 5 & -4 & 2 & 4 \\ 2 & -6 & 10 & -8 & 1 & 5 \\ -3 & 9 & -15 & -12 & 0 & 18 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ has the matrix

$E = \begin{bmatrix} 1 & -3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) Compute:

- $\operatorname{rank} A =$
- $\dim(\ker A) =$
- $\dim(\operatorname{im} A)^{\perp} =$
- $\dim(\ker A)^{\perp} =$

3. 12 pts

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

(b) Use your answer to part (a) to find the projection of $\vec{b}$ onto the image of A.

(c) Determine the error $||\vec{b} - A\vec{x}^*||$.

4. 14 pts Apply the Gram-Schmidt process to the vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

and write the result in the form $A = QR$, with Q orthogonal and R upper-diagonal.

5. 14 pts Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the yz-plane by $45°$ in a counterclockwise direction, and reflects the x-axis about the yz-plane.

(a) Find the matrix A corresponding to T.

(b) What is $\det(A)$?

(c) Is A orthogonal? Why, or why not?

(d) Find A^{-1}.

(e) What is the image of the vector $\begin{bmatrix} 2 \\ \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$ under the map T?

6. 14 pts Let $A = \begin{bmatrix} 5 & 4 & -2 \\ 0 & 3 & 0 \\ 6 & 12 & -3 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

(d) Find an invertible matrix S and a diagonal matrix D such that $A = S \cdot D \cdot S^{-1}$. [You do not have to calculate S^{-1}.]

7. 12 points A 2×2 matrix has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = 3$.

(a) What is the characteristic polynomial of A?

(b) Compute $\operatorname{tr}(A)$ and $\det(A)$.

(c) Compute $\det(5A)$.

(d) What are the eigenvalues of A^2?

(e) Compute $\operatorname{tr}(A^2)$ and $\det(A^2)$.

(f) Is A invertible? If no, say why not. If yes, compute the trace of the inverse matrix.

8. 14 pts Find the singular value decomposition for the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, as follows:

(a) Find the symmetrized matrix $B = A^\top A$.

(b) Find the eigenvalues λ_1 and λ_2 of B.

(c) Find the singular values σ_1 and σ_2 of A.

(d) Find eigenvectors $\vec{w}_1$ and $\vec{w}_2$ for B.

(e) Now find an orthonormal set of eigenvectors, $\vec{v}_1$ and $\vec{v}_2$.

(f) Use the vectors $\vec{v}_i$, the singular values σ_i, and the matrix A to find another orthonormal set, $\vec{u}_1$ and $\vec{u}_2$.

(g) Put everything together to arrive at the SVD decomposition, $A = U \cdot \Sigma \cdot V^\top$. Check your answer!