Lab 3: Using MATLAB for Differential Equations 1

We are now familiar with using a spreadsheet to set up numerical methods for approximating solutions of a differential equation. In this computer lab, we shall not only learn how to use MATLAB to obtain numerical solutions of 1st-order equations of the form $x' = f(t,x)$, but we shall use its algebraic capabilities to obtain general solutions to linear 1st-order systems with constant coefficients.

I. 1st-ORDER EQUATIONS (ode45).

MATLAB has several numerical procedures for computing the solutions of first-order equations and systems of the form $y' = f(t,y)$; we shall concentrate on "ode45", which is a souped-up Runge-Kutta method. The *first step* is to enter the equation by creating an "M-file" which contains the definition of your equation, and is given a name for reference, such as "diffeqn" (the suffix ".m" will be added to identify it as an M-file.). The *second step* is to apply ode45 by using the syntax:

$$[\mathrm{t},\mathrm{y}] = \mathrm{ode45}('\mathrm{diffeqn}', [\mathrm{t_0}, \mathrm{t_f}], \mathrm{y_0}); \tag{1}$$

where $\mathrm{t_0}$ is the initial time, $\mathrm{t_f}$ is the final time, and $\mathrm{y_0}$ is the initial condition, $\mathrm{y(t_0)} = \mathrm{y_0}$. The same syntax (1) works for equations and systems alike.

Example 1. $y' = y^2 - t$, $y(0) = 0$, for $0 \le t \le 4$.

1. *Creating the M-file.* Start up MATLAB; the Command Window appears with the prompt >> awaiting instructions. Choose **New** from the **File** menu, and select **M-file.** You are now in a text editor where you create MATLAB files. Enter the following text:

```
function ypr=example1(t,y)
ypr=y^2-t;
```

Name this M-file "example1.m" by selecting **Save As** from the **File** menu. **Note:** The semicolon at the end tells MATLAB to suppress displaying output. (If you leave out the semicolon and run ode45, MATLAB will display a lot of calculations that you don't need to see.)

2. *Running ode45.* Return to the Command Window, and enter the following:

```
>> [t,y] = ode45('example1',[0,4],0);
```

The $[0,4]$ tells MATLAB to consider $0 \le t \le 4$ and the last 0 tells it to start at $y = 0$. When you hit the enter key, MATLAB will do its computing, then give you another prompt.

3. *Plotting the Solution.* You can plot the solution $y(t)$ by typing

```
>> plot(t,y)
```

and hitting the enter key. To give your plot a title and axes labels, type

```
>>title('The solution to y'' = y^2 - t with y(0) = 0.')
>>xlabel('t')
>>ylabel('y')
```

and hit the enter key after each line. Notice that each title/label is identified by single quotation marks, e.g. 'The solution...'. **Note:** You might expect that the title line should read y' = y^2 − t instead of y'' = y^2 − t, but the former would indicate to MATLAB that the title ends with y', so we must put in the extra single quote (i.e. '' is two single quotes, not one double quote).

You can also have MATLAB tabulate the t-values it has selected and the y-values it has computed by entering

```
>> [t,y]
```

in the Command Window. This should produce a vertical column of numbers, the last of which is $t = 4.0000$ and $y = -1.9311$, i.e. $y(4) = -1.9311$ as appears in the plot.

Exercise 1. Consider the initial value problem $y' = t^2 + \cos y$, $y(0) = 0$ which was encountered in Exercise 4 of Lab 3. Use MATLAB to plot the solution for $0 \le t \le 1$, and find the approximate value of $y(1)$.

→**Hand In:** *A printout of your plot and the value of* $y(1)$.

II. LINEAR 1st-ORDER SYSTEMS (eigenvalues & eigenvectors)

Recall that a first-order system of linear differential equations with constant coefficients may be expressed in matrix notation as

$$\frac{dY}{dt} = AY, \tag{2}$$

where $Y(t)$ is a vector-valued function and A is a square matrix (with constant coefficients). Moreover, if λ_1 is an eigenvalue for A (i.e. $det(A - \lambda_1 I) = 0$) with associated eigenvector V_1 (i.e. $AV_1 = \lambda_1 V_1$), then

$$Y(t) = e^{\lambda_1 t} V_1 \tag{3}$$

is a solution of (2). We shall now use MATLAB to compute the eigenvalues and eigenvectors of a given square matrix A, and therefore calculate the solutions of (2).

The first step is to enter the given matrix A: this is done by enclosing in square brackets the rows of A, separated by semicolons. If we only need the eigenvalues of A, then we can let $E = eig(A)$, and the eigenvalues appear as the column vector E. If we want the eigenvalues and eigenvectors of A, then we can enter $[V, D] = eig(A)$ in order to get two matrices: the matrix V has (unit length) eigenvectors of A as column vectors, and D is a diagonal matrix with the eigenvalues of A on the diagonal.

Example 2. Suppose we want to find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \tag{4}$$

and use them to find the general solution of (2).

Enter the matrix A as follows:

$$>> \texttt{A} = [\texttt{4 2; 1 3}]$$

Now request the eigenvalues of A by entering

$$>> \texttt{E} = \texttt{eig(A)}$$

MATLAB displays the eigenvalues 5 and 2 as the column vector E. Finally, request eigenvectors and eigenvalues of A by entering

$$>> [\texttt{V}, \texttt{D}] = \texttt{eig(A)}.$$

MATLAB displays the following:

$$\texttt{V} =$$
$$\begin{matrix} 0.8944 & -0.7071 \\ 0.4472 & 0.7071 \end{matrix}$$
$$\texttt{D} =$$
$$\begin{matrix} 5 & 0 \\ 0 & 2. \end{matrix}$$

(Actually, 0.8944 may appear as 8.9443e-01, where e-01 means to multiply by 10^{-1}.) The matrix D has the eigenvalues 5 and 2 on the diagonal; the eigenvector corresponding to 5 appears as the first column of the matrix V, namely $V_1 = (0.8944, 0.4472)$. Notice that this is a unit length eigenvector since $(0.8944)^2 + (0.4472)^2 \approx 1$ (with some small round-off error). Since we can multiply both components of an eigenvector by the same number and still get an eigenvector, we could instead take $V_1 = (2, 1)$. Similarly, we could replace the eigenvector $V_2 = (-0.7071, 0.7071)$ corresponding to the eigenvalue 2 by $V_2 = (-1, 1)$.

This means that we have found two linearly independent solutions of (2), $Y_1(t) = e^{5t}\begin{bmatrix}2\\1\end{bmatrix}$ and $Y_2(t) = e^{2t}\begin{bmatrix}-1\\1\end{bmatrix}$, and we can write the general solution as

$$Y(t) = C_1e^{5t}\begin{bmatrix}2\\1\end{bmatrix} + C_2e^{2t}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2C_1e^{5t} - C_2e^{2t}\\ C_1e^{5t} + C_2e^{2t}\end{bmatrix}. \tag{5}$$

If we were given an initial condition $Y(0)$, then we could evaluate C_1 and C_2.

Exercise 2. Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 4x - 2y \\ \frac{dy}{dt} &= x + y. \end{aligned} \tag{6}$$

(a) Letting $Y = \begin{bmatrix}x\\y\end{bmatrix}$, introduce a matrix A so that (6) is in the form (2).

(b) Use MATLAB to determine the eigenvalues and eigenvectors of A.
(c) Use (b) to find two linearly-independent solutions and the general solution of (6).
(d) Use (c) to find the solution of (6) satisfying the initial conditions $x(0) = 1$ and $y(0) = -1$.

Example 3. Suppose we let

$$A = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix}. \tag{7}$$

Proceeding as before, we obtain

$$\mathtt{V} = \begin{bmatrix} .40825 + .40825i & .40825 - .40825i \\ .81650i & -.81650i \end{bmatrix} \qquad \mathtt{D} = \begin{bmatrix} 4+2i & 0 \\ 0 & 4-2i \end{bmatrix}$$

which means that A has complex eigenvalues $\lambda_1 = 4 + 2i$, $\lambda_2 = 4 - 2i$, and associated eigenvectors $V_1 = (1 + i, 2i)$, $V_2 = (1 - i, -2i)$.

This means that one solution of (2) is given by

$$Y_1(t) = e^{(4+2i)t} \begin{bmatrix} 1+i \\ 2i \end{bmatrix} = e^{4t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1+i \\ 2i \end{bmatrix},$$

and the general solution is given by

$$Y(t) = C_1 e^{4t} \begin{bmatrix} \cos 2t - \sin 2t \\ -2\sin 2t \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix}.$$

Given an initial condition $Y(0)$, we could evaluate C_1 and C_2.

Exercise 3. Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= -x - 4y \\ \frac{dy}{dt} &= 3x - 2y. \end{aligned} \tag{8}$$

(a) Use MATLAB to determine the eigenvalues and eigenvectors of the associated matrix.
(b) Use (a) to find two linearly-independent solutions and the general solution of (8).
(c) Use (b) to find the solution of (8) satisfying the initial conditions $x(0) = 1$ and $y(0) = -1$.