1	2	3	4	5	6	7	8	9	Σ

Name: ______________________

NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH U345 # FINAL EXAM **Fall 2008**

Put your name in the blanks above. Put your final answers to each question in the designated spaces. Calculators are permitted. A single sheet of formulas is allowed. **Show your work.**

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1. $\boxed{\text{12 points}}$ Find the general solution to each of the following differential equations.

(a) $y' = \dfrac{y^2}{1+t^2}$

Solution: $y(t) =$ ______________________

(b) $y' = \dfrac{y}{1+t} + t$

Solution: $y(t) =$ ______________________

2. $\boxed{\text{10 points}}$ Newton's Law of Cooling asserts that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the surrounding temperature.

A bottle of of juice at 35°F is placed into a room that has constant temperature of 65°F. Four minutes later, the temperature of the juice is 40°F.

(a) Write down an initial value problem that models the temperature of the juice.

(b) Solve this initial value problem.

(c) Find the temperature of the bottle of juice 10 minutes after it was placed into the room.

3. $\boxed{\text{12 points}}$ Solve the following initial value problems:

(a) $y'' + 4y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$

Solution: $y(t) =$ ____________________

(b) $y'' - 4y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$

Solution: $y(t) =$ ____________________

4. $\boxed{\text{5 points}}$ Convert the following second order differential equation,

$$y'' - 3y' + 4y^2 = 5\sin(t)$$

to a first order system,

$$\begin{cases} y_1' = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \\ y_2' = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$$

How do y_1 and y_2 relate to y? $y_1 =$__________ and $y_2 =$__________.

DO NOT SOLVE the differential equation.

5. $\boxed{\text{9 points}}$ Consider the linear system $Y' = AY$, with $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$.

(a) It turns out that A has a single eigenvalue, λ. Compute this eigenvalue.

(b) Given the initial condition $V_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find the corresponding eigenvector V_1 for the eigenvalue λ.

(c) Find the solution $Y = Y(t)$ with initial condition $Y(0) = V_0$.

6. $\boxed{\text{16 points}}$ Consider the linear system $Y' = AY$, with $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(b) Find (non-zero) eigenvectors V_1 and V_2 corresponding to each eigenvalue.

(c) Find the general solution $Y = Y(t)$ of the given system.

(d) Find the solution $Y = Y(t)$ with initial condition $Y(0) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

(e) Sketch the phase portrait, indicating the straight-line solutions, and at least 4 other solution curves. Put arrows on the curves to show their direction.

(f) What kind of equilibrium point is the origin?

7. $\boxed{\text{10 points}}$ Consider the system

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 3x + 2 \\ \dfrac{dy}{dt} = x^2 - 3y + 5 \end{cases}$$

(a) Find the equilibrium points.

(b) Find the Jacobian matrix of the system.

(c) Find the linearized system for each of the equilibrium points from part (a). In each case, classify the equilibrium point as either source, sink, saddle point, center, etc.

8. $\boxed{\text{12 points}}$

(a) Find the Laplace transform $F(s) = \mathcal{L}[f(t)]$ of the function

$$f(t) = \begin{cases} 0, & t < 1 \\ t^2 + t + 2, & t \geq 1 \end{cases}$$

Solution: $F(s) =$ ______________________

(b) Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the function

$$F(s) = \frac{s-5}{s^2 + 6s + 13}$$

Solution: $f(t) =$ ______________________

9. $\boxed{\text{14 points}}$ Use Laplace transforms to solve the following initial value problems.

(a) $y'' + y = \cos(2t), \quad y(0) = 1, \quad y'(0) = 4$

Solution: $y(t) =$ ____________________________

(b) $y'' + y = \delta_2(t), \quad y(0) = 0, \quad y'(0) = 0$

Solution: $y(t) =$ ____________________________