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MATH 7321 **TOPOLOGY 3** **Fall 2011**

Midterm Exam

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1. Let ΣZ denote the unreduced suspension of a space Z, and let $X * Y$ denote the join of spaces X and Y.

(a) Given a continuous map $\phi\colon X \times Y \to Z$, show that the function $X \times Y \times I \to Z \times I$, $(x, y, t) \mapsto (\phi(x, y), t)$ induces a continuous map $h_\phi\colon X * Y \to \Sigma Z$.

(b) Suppose $\phi\colon X \times X \to X$ is a continuous map such that the maps $L_x\colon X \to X$, $L_x(y) = xy$ and $R_x\colon X \to X$, $R_x(y) = yx$ are homeomorphisms, for all $x \in X$. Show that the map $h_\phi\colon X * X \to \Sigma X$ is a bundle projection, with fiber X.

(c) Use Part (b) to conclude that the Hopf maps $S^3 \to S^2$ and $S^7 \to S^4$ are bundle projections, with fiber S^1 and S^3, respectively.

2. Show that all the Whitehead products in the homotopy groups of an H-space vanish.

3. Let $\iota_n \in \pi_n(S^n)$ be the homotopy class of the identity map.

(a) Show that S^n is an H-space if and only if $[\iota_n, \iota_n] = 0$.

(b) Show that $[\iota_2, \iota_2] = 2\eta$, where η is the generator of $\pi_3(S^2)$ represented by the Hopf map.

4. Let $\alpha \in \pi_1(S^1 \vee S^2)$ and $\beta \in \pi_2(S^1 \vee S^2)$ be represented by the inclusion maps of the factors. Put

$$X = (S^1 \vee S^2) \cup_f D^3,$$

where $f\colon S^2 \to S^1 \vee S^2$ is a map representing $2\beta - \alpha \cdot \beta \in \pi_2(S^1 \vee S^2)$. Show that the inclusion map $i\colon S^1 \to X$ induces isomorphisms $i_\sharp\colon \pi_1(S^1) \xrightarrow{\simeq} \pi_1(X)$ and $i_*\colon H_n(S^1) \xrightarrow{\simeq} H_n(X)$ for all $n \geq 0$, though i is *not* a homotopy equivalence.

5. Let G be an abelian group, and $n > 1$. Show that $H_{n+1}(K(G, n), \mathbb{Z}) = 0$.

6. Let $f\colon X \to Y$ be a map between connected CW-complexes. Show that f is a homotopy equivalence, provided either of the following two conditions holds.

(a) The induced homomorphism $f_\sharp\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism, and f admits a lift $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ to universal covers, such that the induced homomorphism, $\tilde{f}_*\colon H_n(\widetilde{X}) \to H_n(\widetilde{Y})$, is an isomorphism, for all $n \geq 0$.

(b) Both X and Y have dimension at most n, and the induced homomorphism, $f_\sharp\colon \pi_i(X) \to \pi_i(Y)$, is an isomorphism, for all $i \leq n$.

7. Let X be a connected CW-complex with $\pi_n(X) = 0$, for all $n \geq 2$. Show that $\pi_n(X^n)$ is a free abelian group, for all $n \geq 2$.

8. Let G be a group, and let $\{M_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{Z}G$-modules.

(a) Construct a CW-complex X with $\pi_1(X) = G$, and $\pi_n(X) = M_n$ (as $\mathbb{Z}G$-modules).

(b) If $X = K(G,1)\times Y$, where $\pi_1(Y) = 0$, show that $\pi_n(X)$ is trivial as a $\mathbb{Z}G$-module, for all $n > 1$.

9. Let $f = p \circ q\colon T^3 \to S^2$ be the composite of the Hopf map $p\colon S^3 \to S^2$ with the quotient map $q\colon T^3 \to S^3$, collapsing the 2-skeleton of the 3-torus to a point.

(a) Show that $f_* = 0\colon \pi_n(T^3) \to \pi_n(S^2)$, for all $n \geq 1$.

(b) Show that $f_* = 0\colon \widetilde{H}_n(T^3) \to \widetilde{H}_n(S^2)$, for all $n \geq 0$.

(c) Show that, nevertheless, f is *not* homotopic to a constant map.

10. Suppose there exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant 1. Show that n must be a power of 2.