LONG EXACT SEQUENCE IN HOMOLOGY

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Here I will give an overview of the long exact sequence in homology. We could work in an arbitrary abelian category $\mathcal{A}$, but we take $\mathcal{A}$ to be the category $\mathcal{M}od_R$ of R-modules for a ring R.

First we set up what we want to prove. Here we use the notation $X_\bullet$ to denote a chain complex $\{X_n\}_{n\in\mathbf{Z}}$ with boundary maps $d^X = d^X_n : X_n \to X_{n-1}$.

Theorem 0.1 (Long exact sequence in homology). *For a short exact sequence of chain complexes (each in $\mathcal{M}od_R$)*

$$0 \longrightarrow A_\bullet \xrightarrow{\ f\ } B_\bullet \xrightarrow{\ g\ } C_\bullet \longrightarrow 0,$$

there exist natural 'connecting homomorphisms'

$$H_n(C_\bullet) \xrightarrow{\ \partial\ } H_{n-1}(A_\bullet)$$

such that

$$\cdots \xrightarrow{\ \partial\ } H_n(A_\bullet) \xrightarrow{\ f_*\ } H_n(B_\bullet) \xrightarrow{\ g_*\ } H_n(C_\bullet) \xrightarrow{\ \partial\ } H_{n-1}(A_\bullet) \xrightarrow{\ f_*\ } H_{n-1}(B_\bullet) \xrightarrow{\ g_*\ } H_{n-1}(C_\bullet) \xrightarrow{\ \partial\ } \cdots$$

is an exact sequence.

First, we need to define what ∂ is! We use the Snake Lemma, a proof of which we do not provide here (to quote Paolo Aluffi, proving the Snake Lemma is something that should not be done in public).

Lemma 0.2 (Snake Lemma). *For a commutative diagram (in $\mathcal{M}od_R$)*

$$\begin{array}{ccccccc}
 & & A & \longrightarrow & B & \xrightarrow{\ p\ } & C & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\ i\ } & B' & \longrightarrow & C' & &
\end{array}$$

in which the top and bottom rows are exact, there exists an exact sequence

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \xrightarrow{\ \partial\ } \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c,$$

with ∂ a (well-defined) homomorphism

$$\partial(x) := (i^{-1} \circ b \circ p^{-1})(x), \quad \forall x \in \ker c.$$

Now we apply the Snake Lemma! Take the commutative diagram

$$\begin{array}{ccccccc}
 & & A_n/\operatorname{im} d^A_{n+1} & \longrightarrow & B_n/\operatorname{im} d^B_{n+1} & \longrightarrow & C_n/\operatorname{im} d^C_{n+1} & \longrightarrow & 0 \\
 & & {\scriptstyle d^A}\downarrow & & {\scriptstyle d^B}\downarrow & & {\scriptstyle d^C}\downarrow & & \\
0 & \longrightarrow & \ker d^A_n & \xrightarrow{\ f\ } & \ker d^B_n & \xrightarrow{\ g\ } & \ker d^C_n. & &
\end{array} \qquad (\star)$$

We must verify that the top and bottom rows are exact, which follows from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_n^A & \longrightarrow & \ker d_n^B & \longrightarrow & \ker d_n^C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & {\scriptstyle d_n^A}\downarrow & & {\scriptstyle d_n^B}\downarrow & & {\scriptstyle d_n^C}\downarrow & & \\
0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_{n-1}/\operatorname{im} d_n^A & \longrightarrow & B_{n-1}/\operatorname{im} d_n^B & \longrightarrow & C_{n-1}/\operatorname{im} d_n^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Back to the diagram $(\star)$, we have that

$$\ker(A_n/\operatorname{im} d_{n+1}^A \to \ker d_n^A) = H_n(A),$$
$$\operatorname{coker}(A_n/\operatorname{im} d_{n+1}^A \to \ker d_n^A) = H_{n-1}(A),$$

and similarly for the other two columns. Applying the Snake Lemma gives an exact sequence

$$H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{f_*} H_{n-1}(B_\bullet) \xrightarrow{g_*} H_{n-1}(C_\bullet).$$

Via a 'pasting' argument, we obtain the long exact sequence claimed in the Theorem.

Finally, we have naturality.

Proposition 0.3 (Naturality). *Given a commutative diagram of short exact sequences of chain complexes (each in $\mathcal{M}od_R$)*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_\bullet & \xrightarrow{f} & B_\bullet & \xrightarrow{g} & C_\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'_\bullet & \xrightarrow{f'} & B'_\bullet & \xrightarrow{g'} & C'_\bullet & \longrightarrow & 0,
\end{array}$$

there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & H_n(A_\bullet) & \xrightarrow{f_*} & H_n(B_\bullet) & \xrightarrow{g_*} & H_n(C_\bullet) & \xrightarrow{\partial} & H_{n-1}(A_\bullet) & \xrightarrow{f_*} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & H_n(A'_\bullet) & \xrightarrow{f'_*} & H_n(B'_\bullet) & \xrightarrow{g'_*} & H_n(C'_\bullet) & \xrightarrow{\partial} & H_{n-1}(A'_\bullet) & \xrightarrow{f'_*} & \cdots.
\end{array}$$

Proof. As H_n is a functor, the leftmost two squares commute. Next, for $z \in H_n(C_\bullet)$ represented by $c \in C_n$, its image $z' \in H_n(C'_\bullet)$ is represented by the image c' of c. If $b \in B_n$ lifts c, its image in B'_n lifts c'. Hence $\partial z' \in H_{n-1}(A'_\bullet)$ is represented by the image of $d_n^B b \in B_{n-1}$, as $d_n^B b \in \operatorname{im} d_{n-1}^A$. As such, $d_n^B B$ is a representative of $\partial z \in H_{n-1}(A_\bullet)$. Therefore $\partial z'$ is the image of ∂z, and the rightmost square commutes. $\square$

References

[1] Charles A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.