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MATH 4565 **TOPOLOGY** **Spring 2018**

MIDTERM EXAM

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1. Consider the following two topologies on $\mathbb{R}^2$:

(i) The Zariski topology, $\mathcal{T}_1$, where a base of open sets is formed by the complements of zero sets of polynomials in two variables.

(ii) The topology $\mathcal{T}_2$, the weakest topology where all straight lines are closed sets.

Show that $(\mathbb{R}^2, \mathcal{T}_1)$ and $(\mathbb{R}^2, \mathcal{T}_2)$ are not homeomorphic.

2. Define a topology $\mathcal{T}$ on the closed interval $X = [-1, 1]$ by declaring a set to be open if it either does not contain the point 0, or it does contain $(-1, 1)$.

(a) Verify that $\mathcal{T}$ is indeed a topology.

(b) What are the limit points of X?

(c) Is X a T_0 space? (I.e., given a pair of distinct points in X, does at least one of them have a neighborhood not containing the other?)

(d) Is X a T_1 space? (I.e., given a pair of distinct points in X, does each one of them have a neighborhood not containing the other?)

(e) Show that X is compact.

(f) Show that X locally path-connected.

3. Prove or disprove the following:

(a) If X and Y are path-connected, then $X \times Y$ is path-connected.

(b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.

(c) If X is locally path-connected, and $A \subset X$, then A is locally path-connected.

(d) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.

(e) If X is locally path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is locally path-connected.

4. Let $(X, \mathcal{T})$ be a topological space. Show that the following conditions are equivalent:

(a) X is locally connected.

(b) The family of open connected subsets of X forms a basis for $\mathcal{T}$.

5. Let $f\colon X \to Y$ be a continuous map. We say that f is *proper* if $f^{-1}(K)$ is compact, for every compact subset $K \subset Y$. We also say that f is *perfect* if f is surjective, closed, and $f^{-1}(\{y\})$ is compact for every point $y \in Y$.

(a) Show that every continuous map from a compact space to a Hausdorff space is both proper and closed.

(b) Show that every homeomorphism is a perfect map. Conversely, show that every injective perfect map is a homeomorphism.

(c) Give an example of a perfect map which is not open.

6. Let A be a subspace of a topological space X. A *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that A is a *retract* of X.

(a) Prove the following: If X is Hausdorff and A is a retract of X, then A is closed.

(b) By the above, the open interval $(0, 1)$ is *not* a retract of the real line $\mathbb{R}$. Nevertheless, show that the closed interval $[0, 1]$ *is* a retract of $\mathbb{R}$.

7. Let $f\colon X \to Y$ and $g\colon X \to Y$ be two continuous maps. Suppose Y is a Hausdorff space, and that there is a dense subset $D \subset X$ such that $f(x) = g(x)$ for all $x \in D$. Show that $f(x) = g(x)$ for all $x \in X$.

8. Let (X, d) be a metric space, and let $f\colon X \to X$ be a continuous function which has no fixed points.

(a) If X is compact, show that there is a real number $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$, for all $x \in X$.

(b) Show that the conclusion in (a) is false if X is not assumed to be compact.