1. A space $X$ is said to be \textit{homogeneous} if, for every two points $x_1, x_2 \in X$, there is a self-homeomorphism $f : X \to X$ such that $f(x_1) = x_2$. Prove that homogeneity is a topological property. That is to say, if $X$ is homeomorphic to $Y$, and $X$ is homogeneous, then $Y$ is also homogeneous.

2. Let $(X, \mathcal{T})$ be a topological space. Show that the following conditions are equivalent:
   (a) $X$ is locally connected.
   (b) The family of open connected subsets of $X$ is a basis for $\mathcal{T}$.

3. Prove or disprove the following:
   (a) If $X$ and $Y$ are path-connected, then $X \times Y$ is path-connected.
   (b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.
   (c) If $X$ is locally path-connected, and $A \subset X$, then $A$ is locally path-connected.
   (d) If $X$ is path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.
   (e) If $X$ is locally path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is locally path-connected.

4. Let $\mathbb{Z}$ be the set of integers. An \textit{arithmetic progression} is a subset of the form $A_{a,b} = \{a + nb \mid n \in \mathbb{Z}\}$, with $a, b \in \mathbb{Z}$ and $b \neq 0$.
   (a) Prove that the collection of arithmetic progressions,
   \[
   \mathcal{A} = \{A_{a,b} \mid a, b \in \mathbb{Z}, b \neq 0\},
   \]
   is a basis for a topology on $\mathbb{Z}$.
   (b) Is $\mathbb{Z}$ endowed with this topology a Hausdorff space?
   (c) Is $\mathbb{Z}$ endowed with this topology a compact space?
5. Let $f: X \to Y$ be a continuous map. We say that $f$ is proper if $f^{-1}(K)$ is compact, for every compact subset $K \subset Y$. We also say that $f$ is perfect if $f$ is surjective, closed, and $f^{-1}\{y\}$ is compact for every point $y \in Y$.

(a) Show that every continuous map from a compact space to a Hausdorff space is both proper and closed.

(b) Show that every homeomorphism is a perfect map. Conversely, show that every injective perfect map is a homeomorphism.

(c) Give an example of a perfect map which is not open.

6. Let $f: X \to Y$ be a continuous map from a space $X$ to a Hausdorff space $Y$. Let $C$ be a closed subspace of $Y$, and let $U$ be an open neighborhood of $f^{-1}(C)$ in $X$.

(a) Show that if $X$ is compact then there is an open neighborhood $V$ of $C$ in $Y$ such that $f^{-1}(V)$ is contained in $U$.

(b) Give an example to show that if $X$ is not compact, then there need not be such a neighborhood $V$.

7. Let $A$ be a subspace of a topological space $X$. A retraction of $X$ onto $A$ is a continuous map $r: X \to A$ such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that $A$ is a retract of $X$.

(a) Prove the following: If $X$ is Hausdorff and $A$ is a retract of $X$, then $A$ is closed.

(b) By the above, the open interval $(0,1)$ is not a retract of the real line $\mathbb{R}$. Nevertheless, show that the closed interval $[0,1]$ is a retract of $\mathbb{R}$.

8. Let $f: X \to Y$ and $g: X \to Y$ be two continuous maps. Suppose $Y$ is a Hausdorff space, and that there is a dense subset $D \subset X$ such that $f(x) = g(x)$ for all $x \in D$. Show that $f(x) = g(x)$ for all $x \in X$. 