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MATH 4565 **TOPOLOGY** **Fall 2021**

Handout 2: Interior, closure, and boundary

Let X be a topological space. For a subset $A \subset X$, we define the *interior* and the *closure* of A in X to be the sets

$$\mathrm{Int}(A) := \bigcup_{\substack{U \subset A \\ U \text{ open}}} U, \tag{1}$$

$$\overline{A} := \bigcap_{\substack{C \supset A \\ C \text{ closed}}} C. \tag{2}$$

Furthermore, the *exterior* of A is the complement in X of the closure of A, while the *boundary* of A is the complement in X of the union of the interior and the exterior of A,

$$\mathrm{Ext}(A) := X \setminus \overline{A}, \qquad \partial A := X \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A)). \tag{3}$$

The following results give necessary and sufficient conditions for a point in X to belong to either the interior, closure, or boundary of A.

Proposition 1. *A point $x \in X$ belongs to the interior of A if and only if there is a neighborhood of x contained in A.*

Proof. By definition (1), $x \in \mathrm{Int}(A)$ if and only if x belongs to some open subset U which is contained in A; in other words, if and only if there is an (open) neighborhood U of x such that $U \subset A$. □

Note the contrapositive formulation of this proposition: A point $x \in X$ does not belong to the interior of A if and only if any neighborhood U of x contains a point y which is not in A, that is, $U \not\subset A$.

Proposition 2. *For a point $x \in X$, the following are equivalent.*

(i) $x \in \overline{A}$.

(ii) If U is a neighborhood of x, then $U \cap A \neq \emptyset$.

Proof. (i)⇒(ii) By definition (2), a point x belongs to $\overline{A}$ if and only if, for all closed subsets C that contain A, we have that $x \in C$. Now let U be an open subset of X that contains x and suppose $U \cap A = \emptyset$. Then the complement $U^{\complement} = X \setminus U$ is a closed subset and $A \subset U^{\complement}$. Hence, $x \in U^{\complement}$, contradicting our assumption that $x \in C$. Therefore, $U \cap A \neq \emptyset$.

(ii)⇒(i) Suppose that $U \cap A \neq \emptyset$ for all open subsets U such that $x \in U$. Let C be a closed subset such that $C \supset A$, and suppose $x \notin C$. Then the complement $C^{\complement} = X \setminus C$ is an open subset containing x, and so $C^{\complement} \cap A \neq \emptyset$, thereby contradicting our assumption that $A \subset C$. Therefore, $x \in C$, and the proof is complete. □

Note the contrapositive formulation of this proposition: A point $x \in X$ does not belong to the closure of A if and only if there is a neighborhood of x which does not intersect A.

Proposition 3. *A point $x \in X$ belongs to the boundary of A if and only if every neighborhood of x contains both a point of A and a point of $X \setminus A$.*

Proof. By De Morgan's Laws and the definitions of the boundary and exterior of A, we have that

$$\begin{aligned}\partial A &= X \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A)) \\ &= (X \setminus \mathrm{Int}(A)) \cap (X \setminus \mathrm{Ext}(A)) \\ &= (X \setminus \mathrm{Int}(A)) \cap (X \setminus (X \setminus \overline{A})) \\ &= (X \setminus \mathrm{Int}(A)) \cap \overline{A}.\end{aligned}$$

Thus, $x \in \partial A$ if and only if $x \in X \setminus \mathrm{Int}(A)$ and $x \in \overline{A}$.

So assume $x \in \partial A$, and let U be a neighborhood of x. Since x is not contained in $\mathrm{Int}(A)$, Proposition 1 implies that U is not contained in A; that is, there is a point $y \in U$ such that $y \in X \setminus A$. On the other hand, since $x \in \overline{A}$, Proposition 2 implies that $U \cap A \neq \emptyset$, that is, there is a point $z \in U$ such that $z \in A$.

The converse statement is proved by tracing back through the above argument. This completes the proof. □

Note the contrapositive formulation of this proposition: A point $x \in X$ does not belong to the boundary of A if and only if every neighborhood of x is disjoint from both A and its complement.