Prof. Alexandru Suciu

MATH 3175 | **Group Theory** | **Summer 2, 2022**

Practice Problems for Review 1

---

1. Prove, by comparing orders of elements, that the following pairs of groups are not isomorphic:

(i) $\mathbb{Z}_8 \times \mathbb{Z}_8$ and $\mathbb{Z}_{16} \times \mathbb{Z}_4$.

(ii) $\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3$ and $\mathbb{Z}_{27} \times \mathbb{Z}_9$.

2. Describe a specific isomorphism $\varphi\colon \mathbb{Z}_6 \times \mathbb{Z}_5 \to \mathbb{Z}_{30}$.

3. Describe a specific isomorphism $\psi\colon \mathbb{Z}_{16}^{\times} \to \mathbb{Z}_2 \times \mathbb{Z}_4$.

4. How many elements of order 6 are there in $\mathbb{Z}_6 \times \mathbb{Z}_9$?

5. How many elements of order 25 are there in $\mathbb{Z}_5 \times \mathbb{Z}_{25}$?

6. Let p be a prime. Determine the number of elements of order p in $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.

7. Let $G = S_3 \times \mathbb{Z}_5$. What are all possible orders of elements in G? Prove that G is *not* cyclic.

8. Let H be set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, b, d \in \mathbb{R}$ and $ad \neq 0$.

(i) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(ii) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$?

9. Let H be set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, with $a, c, d \in \mathbb{Z}$ and $ad = \pm 1$.

(i) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

(ii) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{Z})$?

10. Let $H = \{(1), (12)(34)\}$.

(i) Show that H is a subgroup of A_4.

(ii) What is the index of H in A_4?

(iii) Is H a normal subgroup of A_4?

11. Let $G = \mathbb{Z}_{32}^{\times}$, and $H = \{1, 31\}$. Show that the quotient group G/H is isomorphic to $\mathbb{Z}_8$.

12. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4^{\times}$, and consider the subgroups $H = \langle (2,3) \rangle$ and $K = \langle (2,1) \rangle$.

(i) List the elements of G/H, and compute the Cayley table for this group. What is the isomorphism type of G/H?

(ii) List the elements of G/K, and compute the Cayley table for this group. What is the isomorphism type of G/K?

(iii) Are the groups G/H and G/K isomorphic?

13. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, and consider the subgroups $H = \{(0,0),(2,0),(0,2),(2,2)\}$ and $K = \langle(1,2)\rangle$. Identify the following groups (as direct products of cyclic groups of prime order):

(i) H and G/H.

(ii) K and G/K.

14. Give an example of a group G and a normal subgroup $H \triangleleft G$ such that both H and G/H are abelian, yet G is not abelian.

15. Let $\mathbb{Z}$ be the additive group of integers, and let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = 8x$.

(i) Show that f is a homomorphism.

(ii) Find $\ker(f)$.

(iii) Find $\operatorname{im}(f)$.

16. Let $\varphi\colon G \to H$ and $\psi\colon H \to K$ be two homomorphisms.

(i) Show that $\psi \circ \varphi\colon G \to K$ is a homomorphism.

(ii) Show that $\ker(\varphi)$ is a normal subgroup of $\ker(\psi \circ \varphi)$.

17. Let G and H be two groups, and consider the map $p\colon G \times H \to H$ given by $p(g,h) = h$.

(i) Show that p is a homomorphism.

(ii) What is $\ker(p)$? What is $\operatorname{im}(p)$?

(iii) What does the First Isomorphism Theorem say in this situation?

18. Let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{R}^\times$ be the multiplicative group of non-zero real numbers. Consider the map $\varphi\colon \mathbb{R} \to \mathbb{R}^\times$ given by $\varphi(x) = e^x$.

(i) Show that φ is an homomorphism from $\mathbb{R}$ to $\mathbb{R}^\times$.

(ii) What is the kernel of φ?

(iii) What is the image of φ? For each $y \in \operatorname{im}(\varphi)$ find an $x \in \mathbb{R}$ such that $\varphi(x) = y$.

(iv) Is φ injective?

(v) Is φ surjective?

(vi) Is φ an isomorphism?

19. Let $\varphi\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ be the map given by $\varphi(x) = \begin{cases} 1 & \text{if } x = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise.} \end{cases}$

(i) Show that φ is a homomorphism.

(ii) What is $\ker(\varphi)$? What is $\operatorname{im}(\varphi)$?

20. Suppose $\varphi\colon \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a homomorphism with $\varphi(7) = 6$.

(i) Determine $\varphi(x)$, for all $x \in \mathbb{Z}_{50}$.

(ii) What is $\ker(\varphi)$? What is $\operatorname{im}(\varphi)$?

(iii) What is $\varphi^{-1}(3)$?

21. Show that there is no homomorphism from $\mathbb{Z}_8 \times \mathbb{Z}_2$ onto $\mathbb{Z}_4 \times \mathbb{Z}_4$.

22. Find $\operatorname{Aut}(S_3)$.

23. Find $\operatorname{Aut}(Q_8)$.

24. Let G be a group.

(i) Show that, if G is abelian, then any subgroup of G is normal.

(ii) Is the intersection of a collection of normal subgroups of G normal?

(iii) Let $K \leq H \leq G$ be subgroups of G, and suppose that K is normal in G. Is $K normal in H$?

(iv) Let $K \leq H \leq G$ be subgroups of G, and suppose that K is normal in H. Is K normal in G?

25. Let $f\colon G \to H$ be a function between two groups, and let

$$K := \{(x, y) \in G \times H \mid f(x) = y\}$$

be its graph. Show that f is a homomorphism if and only if K is a subgroup of the direct product $G \times H$.

26. Let G be a group with center $Z(G)$, and let H be a subgroup of G.

(i) Show that if $H \subset Z(G)$, then H is normal in G.

(ii) Show that if $H \subset Z(G)$ and G/H is cyclic, then G is abelian.

27. Let $G = \mathrm{GL}_3(\mathbb{Z}_2)$ be the group of invertible 3×3 matrices with entries in $\mathbb{Z}_2$. Find a subgroup $H \leq G$ of order 8.

28. Let A_4 be the group of even permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroups $H = \langle(123)\rangle$ and $K = \langle(12)(34)\rangle$.

(i) Write down all the **left** and **right** cosets of H in A_4. Be sure to indicate the elements of each coset.

(ii) What is the order of H? What is the index of H in A_4? Is H a normal subgroup of A_4?

(iii) Write down all the **left** and **right** cosets of K in A_4. Be sure to indicate the elements of each coset.

(iv) What is the order of K? What is the index of K in A_4? Is K a normal subgroup of A_4?

(v) Find the intersection $H \cap K$. Is this a subgroup of A_4? Is this a normal subgroup of A_4?

(vi) Find the direct product $H \times K$ and identify it (up to isomorphism) as another well-known group of the same order.