Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Fall 2010**

Solutions to Quiz 5

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1. List all the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, and compute their orders.

Element (a,b)	(0,0)	(1,0)	(0,1)	(1,1)	(0,2)	(1,2)	(0,3)	(1,3)
order $\|(a,b)\|$	1	2	8	8	4	4	8	8
Element (a,b)	(0,4)	(1,4)	(0,5)	(1,5)	(0,6)	(1,6)	(0,7)	(1,7)
order $\|(a,b)\|$	2	2	8	8	4	4	8	8

2. Show that the group $U(9)$ is isomorphic to the direct product $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, by describing *explicitly* an isomorphism $\phi\colon U(9) \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$$\begin{aligned} U(9) &\to \mathbb{Z}_2 \oplus \mathbb{Z}_3 \\ 1 &\mapsto (0,0) \\ 2 &\mapsto (1,1) \\ 4 &\mapsto (0,2) \\ 5 &\mapsto (1,2) \\ 7 &\mapsto (0,1) \\ 8 &\mapsto (1,0) \end{aligned}$$

3. Consider the group $G = S_3 \oplus \mathbb{Z}_6$.

(a) Determine the set of orders of elements in G, that is, the set $\{|g| \mid g \in G\}$.

Orders of elements in S_3: 1, 2, 3;
Orders of elements in $\mathbb{Z}_6$: 1, 2, 3, 6;
Orders of elements in $S_3 \oplus \mathbb{Z}_6$: 1, 2, 3, 6.

(b) Prove that G is *not* cyclic.

The order of G is 36, but there are no elements of order 36 in G. Hence G is not cyclic.

4. How many elements of order 7 are there in $\mathbb{Z}_{70} \oplus \mathbb{Z}_{490}$?

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\begin{aligned}
\#\{\text{elements of order 7 in } \mathbb{Z}_{70}\} &= \phi(7) = 6 \\
\#\{\text{elements of order 7 in } \mathbb{Z}_{490}\} &= \phi(7) = 6 \\
\#\{\text{of elements of order 7 in } \mathbb{Z}_{70} \oplus \mathbb{Z}_{490}\} &= \phi(7) \times 1 + \phi(7) \times \phi(7) + 1 \times \phi(7) \\
&= 6 + 6 \times 6 + 6 = 48
\end{aligned}
$$

5. List all abelian groups (up to isomorphism) of order 72. Write each such group as a direct product of cyclic groups of prime power order.

$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2}$
$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

WARNING: Since the problem asks you to write down the groups as direct products of cyclic groups of prime power order, you cannot write for example $\mathbb{Z}_{72}$ instead of $\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$, even though the two groups are isomorphic (because 2^3 is prime to 3^2).

6. Let G be an abelian group of order 108. Suppose that G has exactly eight elements of order 3, and one element of order 2. Determine the isomorphism class of G.

The order of G has prime factorization $108 = 2^2 \times 3^3$.
The abelian groups of order 108 (up to isomorphism) are:

$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^3}$
$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_3$ (**This group satisfies the conditions.**)
$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ (This group has 26 elements of order 3 and 1 element of order 2.)
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^3}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$