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MATH 3175 | **Group Theory** | **Fall 2010**

Solutions to Quiz 4

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1. (5 points) Let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Consider the map $\phi\colon \mathbb{R} \to \mathbb{R}^+$ given by $\phi(x) = 2^x$.

(a) Show that ϕ is an isomorphism from $\mathbb{R}$ to $\mathbb{R}^+$.

We need to show that ϕ is a bijection, and a homomorphism.

- ϕ injective. Suppose $2^x = 2^y$. Taking $\log_2$ on both sides, we get $x = y$.
- ϕ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = \log_2 y$.
- ϕ a homomorphism. Compute: $\phi(x+y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$.

(b) What is the inverse isomorphism?

$\phi^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$, given by $\phi^{-1}(y) = \log_2 y$

2. (4 points) Show that the automorphism group $\mathrm{Aut}(\mathbb{Z}_{10})$ is isomorphic to a cyclic group $\mathbb{Z}_n$. What is n?

$\mathrm{Aut}(\mathbb{Z}_{10}) \cong U(10) \cong \mathbb{Z}_4$

3. (6 points) Show that the following pairs of groups are *not* isomorphic. In each case, explain why.

(a) $U(12)$ and $\mathbb{Z}_4$.

$U(12)$ is not cyclic, since $|U(12)| = 4$, but $U(12)$ has no element of order 4. On the other hand, $\mathbb{Z}_4$ is cyclic. Thus, $U(12) \ncong \mathbb{Z}_4$.

(b) S_3 and $\mathbb{Z}_6$.

S_3 is not abelian, since, for instance, $(12)\cdot(13) \neq (13)\cdot(12)$. On the other hand, $\mathbb{Z}_6$ is abelian (all cyclic groups are abelian.) Thus, $S_3 \ncong \mathbb{Z}_6$.

(c) S_4 and D_{12}.

Each permutation of S_4 can be written as composition of disjoint cycles. So the only possible orders for the elements in S_4 are 1, 2, 3, and 4. On the other hand, there is an element of order 12 in D_{12}, for instance, the counter-clockwise rotation by 30 degrees. Thus, $S_4 \ncong D_{12}$.

4. (5 points) Let G be a group, and let a be an element of order 30. How many left cosets of $\langle a^6\rangle$ in $\langle a\rangle$ are there? List all these cosets. (Make sure to indicate all the elements in each coset.)

Solution.

$|\langle a\rangle| = |a| = 30$, $|\langle a^6\rangle| = |a^6| = 30/\gcd(6, 30) = 5$.

The number of cosets is $|\langle a\rangle| \,/\, \left|\langle a\rangle\right| = 30/5 = 6$. They are

$$\begin{aligned}
\langle a\rangle &= \{e, a^6, a^{12}, a^{18}, a^{24}\}\\
a\langle a\rangle &= \{a, a^7, a^{13}, a^{19}, a^{25}\}\\
a^2\langle a\rangle &= \{\cdots\}\\
a^3\langle a\rangle &= \{\cdots\}\\
a^4\langle a\rangle &= \{\cdots\}\\
a^5\langle a\rangle &= \{\cdots\}
\end{aligned}$$

5. (5 points) Let S_3 be the group of permutations of the set $\{1, 2, 3\}$. Consider the subgroup $H = \{(1), (13)\}$.

(a) Write down all the left cosets of H in S_3. (Make sure to indicate all the elements in each coset.)

$$\begin{aligned}
H &= \{(1), (1\ 3)\}\\
(1\ 2)H &= \{(1\ 2), (1\ 3\ 2)\}\\
(1\ 2\ 3)H &= \{(1\ 2\ 3), (2\ 3)\}
\end{aligned}$$

(b) What is the index of H in S_3?

$[S_3 : H] = |S_3| \,/\, |H| = 6/2 = 3$

6. (5 points) Suppose G is a group of order 35.

(a) What are the possible orders for the elements of G?

According to Lagrange's theorem, the order of an element in G divides the order of the group. So the possible orders of elements are 1, 5, 7 and 35.

(b) Suppose G has an element of order 35. What is G?

G must be cyclic, and so $G = \mathbb{Z}_{35}$.

(c) Suppose G has precisely one subgroup of order 5, and one subgroup of order 7. What is G?

Again, $G = \mathbb{Z}_{35}$.

Indeed, suppose the subgroup of order 5 is H, and the one of order 7 is K. Then $H \cup K$ has

$$1 + (5 - 1) + (7 - 1) = 11$$

elements. Choose an element a from $G \setminus (H \cup K)$. The order of a is 1, 5, 7 or 35.

We claim the order of a is 35. The order of a is not 1 because it is not the identity. The order of a is neither 5 nor 7, for otherwise a would generate a subgroup $\langle a \rangle$ of order 5 or 7, distinct from H or K, and we know there is precisely one subgroup of order 5 (namely, H), and precisely one subgroup of order 7 (namely, K).

Thus $G = \langle a \rangle$, and so G is cyclic.