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MATH 3175 **Group Theory** **Fall 2010**

Solutions to Quiz 1

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1. Consider the integers $a = 18$ and $b = 27$.

(i) Find $d = \gcd(18, 27)$ and $\ell = \operatorname{lcm}(18, 27)$.

$$d = 9, \quad \ell = 54$$

(ii) What is the relationship between a, b, d, and ℓ predicted by the general theory? Verify this relationship holds in this situation.

$$d \cdot \ell = a \cdot b : \quad 9 \cdot 54 = 18 \cdot 27 = 486.$$

(iii) Find a pair of integers s and t such that $18s + 27t = d$.

$$18 \cdot (-1) + 27 \cdot 1 = 9 \implies s = -1, t = 1$$

(iv) Find the general solution for all the pairs of integers s and t such that $18s + 27t = d$.

$$s = -1 - 3k, t = 1 + 2k$$

2. The following Latin square is the Cayley table of a group:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(i) Verify associativity for the non-identity elements in the group.

$$(ab)c = cc = e \qquad a(bc) = aa = e$$
$$(aa)b = eb = b \qquad a(ab) = ac = b$$
$$(aa)a = ea = a \qquad a(aa) = ae = a$$

(ii) Is the group abelian? Why, or why not?

The Cayley table is symmetric. Thus, the group is abelian.

(iii) What are the inverses of a, b, and c, respectively?

$$a^{-1} = a, \quad b^{-1} = b, \quad c^{-1} = c.$$

(iv) Is the inverse of ab equal to ba? Why, or why not?

$$(ab)^{-1} = b^{-1}a^{-1} = ba$$

3. Show that the following identities hold in any group. Explain your reasoning.

(i) $(a^{-1})^{-1} = a$. The fact that a^{-1} is the inverse of a is expressed as:

$$a^{-1} \cdot a = a \cdot a^{-1} = e.$$

But this also means a is the inverse of a^{-1}, i.e., $a = (a^{-1})^{-1}$.

(ii) $(a^{-1}ba)^3 = a^{-1}b^3a$.

$$(a^{-1}ba)^3 = a^{-1}ba \cdot a^{-1}ba \cdot a^{-1}ba = a^{-1}bebeba \cdot a^{-1} = a^{-1}b^3a.$$

4. Consider the group $U(12)$.

(i) List all the elements in $U(12)$, and write down the Cayley table for the group.

$$U(12) = \{1, 5, 7, 11\}$$

	1	5	7	11
1	1	5	7	11
5	5	1	11	5
7	7	11	1	5
11	11	5	5	1

(ii) For each element a in $U(12)$, indicate what is a^{-1}.

$$1^{-1} = 1, \quad 5^{-1} = 5, \quad 7^{-1} = 7, \quad 11^{-1} = 11.$$

5. Consider the following two matrices, viewed as elements in the group $\mathrm{GL}_2(\mathbb{Z}_7)$:

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Find the inverse of the A in $\mathrm{GL}_2(\mathbb{Z}_7)$.

$$\det A = 4 \cdot 2 - 1 \cdot 3 = 5 \qquad (\det A)^{-1} = 5^{-1} = 3.$$

$$A^{-1} = 3 \cdot \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -9 & 12 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 5 & 5 \end{pmatrix}$$

(ii) Compute the products $A \cdot B$ and $B \cdot A$. Are they the same, or not?

$$A \cdot B = \begin{pmatrix} 4 & 5 \\ 3 & 5 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$$

Thus, the two products are different.

(iii) Is the group $\mathrm{GL}_2(\mathbb{Z}_7)$ commutative? Why, or why not?

No, the group is not commutative, since it has a pair of elements (the matrices A an B above) which do not commute.

6. Let G a group such that $(ab)^{-1} = a^{-1}b^{-1}$, for all a and b in G. Prove that G is abelian.

Let a and b be two elements in G. We then always have

$$(ab)^{-1} = b^{-1}a^{-1}$$

In our situation, we also have

$$(ab)^{-1} = a^{-1}b^{-1}$$

Thus,

$$b^{-1}a^{-1} = a^{-1}b^{-1}.$$

Now take inverses on both sides:

$$(b^{-1}a^{-1})^{-1} = (a^{-1}b^{-1})^{-1}.$$

Using the first formula, together with the identity from Problem 3(i), we get:

$$ab = ba$$

We have shown that any pair of elements in G commutes. Thus, G is abelian.