1. Prove, by comparing orders of elements, that the following pairs of groups are not isomorphic:

   (a) $\mathbb{Z}_8 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$.
   
   There is an element of order 16 in $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$, for instance, $(1, 0)$, but no element of order 16 in $\mathbb{Z}_8 \oplus \mathbb{Z}_4$.

   (b) $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$.
   
   There is an element of order 27 in $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$, for instance, $(1, 0)$, but no element of order 27 in $\mathbb{Z}_9 \oplus \mathbb{Z}_9$.

2. Describe a specific isomorphism $\phi: \mathbb{Z}_6 \oplus \mathbb{Z}_5 \rightarrow \mathbb{Z}_{30}$.

   Set $\phi((1, 1)) = 1$, and then use the fact that $\phi$ is a homomorphism to determine $\phi((i, j))$.

3. Describe a specific isomorphism $\psi: U(16) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

   
   \[
   \begin{align*}
   1 & \mapsto (0, 0) \\
   3 & \mapsto (0, 1) \\
   5 & \mapsto (1, 1) \\
   7 & \mapsto (1, 0) \\
   9 & \mapsto (0, 2) \\
   11 & \mapsto (0, 3) \\
   13 & \mapsto (1, 3) \\
   15 & \mapsto (1, 2)
   \end{align*}
   \]

4. Prove or disprove that $D_6 \cong D_3 \oplus \mathbb{Z}_2$.

   Yes, the two groups are isomorphic. Why?

5. Prove or disprove that $D_{12} \cong D_4 \oplus \mathbb{Z}_3$.

   Hint: count elements of order 2

6. How many elements of order 6 are there in $\mathbb{Z}_6 \oplus \mathbb{Z}_9$?

   The order of $(a, b)$ is the least common multiple of the order of $a$ and that of $b$. We would like the order of $(a, b)$ to be 6. This can happen only if the order of $a$ is 6 and that of $b$ is 1 or 3, or the order of $a$ is 2 and that of $b$ is 3. The desired elements of order 6 are:

   $(1, 0), (5, 0), (1, 3), (1, 6), (5, 3), (5, 6), (3, 3), (3, 6)$
7. How many elements of order 25 are there in $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$?

The number of elements of order 25 in $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ equals

$$1 \times \phi(25) + \phi(5) \times \phi(25) = (25 - 5) + (5 - 1) \times (25 - 5) = 100.$$ 

Note 1: The number of elements of order 5 equals $\phi(25) + \phi(5) = (25 - 5) + (5 - 1) = 24$. Accounting also for the single element of order 1, namely the identity $(0, 0)$, we have in all $100 + 24 + 1 = 125$ elements $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$, as we should (check: $5 \cdot 25 = 125$).

Note 2: We used here the fact that $\phi(p^n) = p^n - p^{n-1}$ for any odd prime $p$, which follows from the corresponding fact about $U(p^n)$ mentioned in the solution to Problem 13 below.

8. How many elements of order 3 are there in $\mathbb{Z}_{300000} \oplus \mathbb{Z}_{900000}$?

$$1 \times \phi(3) + \phi(3) \times \phi(3) + \phi(3) \times 1 = 8$$

9. Let $p$ be a prime. Determine the number of elements of order $p$ in $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$.

$$1 \times \phi(p) + \phi(p) \times \phi(p) + \phi(p) \times 1 = p^2 - 1$$

10. Let $G = S_3 \oplus \mathbb{Z}_5$. What are all possible orders of elements in $G$? Prove that $G$ is not cyclic.

Possible orders: 1, 2, 3, 5, 10, 15

The order of $G$ is 30. There is no element of order 30 in the group, so $G$ is not cyclic.

11. The group $S_3 \oplus \mathbb{Z}_2$ is isomorphic to one of the following groups: $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \oplus \mathbb{Z}_2$, $A_4$, $D_6$. Determine which one, by a process of elimination.

The group $S_3 \oplus \mathbb{Z}_2$ is not abelian, but $\mathbb{Z}_{12}$ and $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ are.

The elements of $S_3 \oplus \mathbb{Z}_2$ have order 1, 2, 3, or 6, whereas the elements of $A_4$ have order 1, 2, or 3.

So what’s the conclusion?

12. Describe all abelian groups of order 1,008 = $2^4 \cdot 3^2 \cdot 7$. Write each such group as a direct product of cyclic groups of prime power order.

$\mathbb{Z}_{2^4} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_{2^4} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$.
13. Describe $U(1008)$ as a direct product of cyclic groups.

Some general facts worth knowing:

\[
\begin{align*}
U(m \cdot n) & \cong U(m) \oplus U(n) & \text{if } \gcd(m, n) = 1 \\
U(2) & \cong \{0\}, \quad U(4) \cong \mathbb{Z}_2 \\
U(2^n) & \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-2}} & \text{for all } n \geq 3 \\
U(p^n) & \cong \mathbb{Z}_{p^n - p^{n-1}} & \text{for any odd prime } p
\end{align*}
\]

Hence:

\[
U(1008) \cong U(2^4) \oplus U(3^2) \oplus U(7) \\
\cong (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6 \\
\cong \mathbb{Z}_2^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6^2
\]

14. Describe $U(195)$ as a direct product of cyclic groups in four different ways.

\[
U(195) \cong U(3) \oplus U(5) \oplus U(13) \\
\cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \\
\cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \\
\cong \mathbb{Z}_6 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4
\]

15. For each of the following groups, compute the number of elements of order 1, 2, 4, 8, and 16:

\[
\begin{array}{|c|cccc|}
\hline
\text{Group} & \text{Order} & 1 & 2 & 4 & 8 & 16 \\
\hline
\mathbb{Z}_{16} & & 1 & 1 & 2 & 4 & 8 \\
\mathbb{Z}_8 \oplus \mathbb{Z}_2 & & 1 & 3 & 4 & 8 & 0 \\
\mathbb{Z}_4 \oplus \mathbb{Z}_4 & & 1 & 3 & 12 & 0 & 0 \\
\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & 1 & 7 & 8 & 0 & 0 \\
\hline
\end{array}
\]

16. List all abelian groups (up to isomorphism) of order 160 = $2^5 \cdot 5$.

\[
\begin{align*}
\mathbb{Z}_{2^5} \oplus \mathbb{Z}_5 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_{2^4} \oplus \mathbb{Z}_5 \\
\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_5 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_5 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_5 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5
\end{align*}
\]
16’. List all abelian groups (up to isomorphism) of order $360 = 2^3 \cdot 3^2 \cdot 5$.

- $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{360}$
- $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{180} \oplus \mathbb{Z}_2$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{90} \oplus \mathbb{Z}_2^2$
- $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_3$
- etc

17. (a) List the five partitions of 4, and the abelian groups of order 81 that correspond to them.

$4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1$

$\mathbb{Z}_{81}, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_{27}, \quad \mathbb{Z}_9 \oplus \mathbb{Z}_9, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

(b) A certain abelian group $G$ of order 81 has no elements of order 27, and 54 elements of order 9. Which group is it? Why?

$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$

18. How many abelian groups (up to isomorphism) are there

(a) of order 21? One: $\mathbb{Z}_{21}$
(b) of order 105? One: $\mathbb{Z}_{105}$
(c) of order 210? One: $\mathbb{Z}_{210}$
(d) of order 25? $25 = 5 \times 5$, so there are two, $\mathbb{Z}_{25}$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_5$
(e) of order 125? Use: $125 = 5 \times 25 = 5 \times 5 \times 5$
(f) of order 625? Use: $625 = 5 \times 125 = 25 \times 25 = 5 \times 5 \times 25 = 5 \times 5 \times 5 \times 5$

19. Let $G$ be a finite abelian group of order $n$.

(a) Suppose $n$ is divisible by 10. Show that $G$ has a cyclic subgroup of order 10.

According to the decomposition theorem for finite abelian groups, $G$ contains the group $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ as a subgroup, which is cyclic of order 10.

(b) Suppose $n$ is divisible by 9. Show, by example, that $G$ need not have a cyclic subgroup of order 9.

Take $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

20. Suppose $G$ is an abelian group of order 168, and that $G$ has exactly three elements of order 2. Determine the isomorphism class of $G$.

$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7$. 