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MATH 3175 **Group Theory** **Summer 2, 2022**

The dihedral groups

The general setup. The dihedral group D_n is the group of symmetries of a regular polygon with n vertices. We think of this polygon as having vertices on the unit circle, with vertices labeled $0, 1, \dots, n-1$ starting at $(1, 0)$ and proceeding counterclockwise at angles in multiples of $360/n$ degrees, that is, $2\pi/n$ radians.

There are two types of symmetries of the n-gon, each one giving rise to n elements in the group D_n:

- Rotations $R_0, R_1, \dots, R_{n-1}$, where R_k is rotation of angle $2\pi k/n$.
- Reflections $S_0, S_1, \dots, S_{n-1}$, where S_k is reflection about the line through the origin and making an angle of $\pi k/n$ with the horizontal axis.

The group operation is given by composition of symmetries: if a and b are two elements in D_n, then $a \cdot b = b \circ a$. That is to say, $a \cdot b$ is the symmetry obtained by applying first a, followed by b.

The elements of D_n can be thought as linear transformations of the plane, leaving the given n-gon invariant. This lets us represent the elements of D_n as 2×2 matrices, with group operation corresponding to matrix multiplication. Specifically,

$$R_k = \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ \sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix},$$

$$S_k = \begin{pmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ \sin(2\pi k/n) & -\cos(2\pi k/n) \end{pmatrix}.$$

It is now a simple matter to verify that the following relations hold in D_n:

$$\begin{aligned} R_i \cdot R_j &= R_{i+j} \\ R_i \cdot S_j &= S_{i+j} \\ S_i \cdot R_j &= S_{i-j} \\ S_i \cdot S_j &= R_{i-j} \end{aligned}$$

where $0 \le i, j \le n-1$, and both $i+j$ and $i-j$ are computed modulo n.

The Cayley table for D_n can be readily computed from the above relations. In particular, we see that R_0 is the identity, $R_i^{-1} = R_{n-i}$, and $S_i^{-1} = S_i$.

The group D_3. This is the symmetry group of the equilateral triangle, with vertices on the unit circle, at angles 0, $2\pi/3$, and $4\pi/3$. The matrix representation is given by

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad R_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

while the Cayley table for D_3 is:

	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0	R_1	R_2	S_0	S_1	S_2
R_1	R_1	R_2	R_0	S_1	S_2	S_0
R_2	R_2	R_0	R_1	S_2	S_0	S_1
S_0	S_0	S_2	S_1	R_0	R_2	R_1
S_1	S_1	S_0	S_2	R_1	R_0	R_2
S_2	S_2	S_1	S_0	R_2	R_1	R_0

The group D_4. This is the symmetry group of the square with vertices on the unit circle, at angles 0, $\pi/2$, π, and $3\pi/2$. The matrix representation is given by

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

while the Cayley table for D_4 is:

	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_0	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_1	R_1	R_2	R_3	R_0	S_1	S_2	S_3	S_0
R_2	R_2	R_3	R_0	R_1	S_2	S_3	S_0	S_1
R_3	R_3	R_0	R_1	R_2	S_3	S_0	S_1	S_2
S_0	S_0	S_3	S_2	S_1	R_0	R_3	R_2	R_1
S_1	S_1	S_0	S_3	S_2	R_1	R_0	R_3	R_2
S_2	S_2	S_1	S_0	S_3	R_2	R_1	R_0	R_3
S_3	S_3	S_2	S_1	S_0	R_3	R_2	R_1	R_0