Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Summer 2, 2022**

Assignment 4

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1. Let S_n be the symmetric group on n elements, let $A_n \leq S_n$ be the alternating subgroup, consisting of all even permutations). Given a subgroup $H \leq S_n$, show that either

(a) every permutation in H is even; or

(b) the set $H \cap A_n$ is properly contained in H, and, moreover, half the permutations in H are even and half are odd.

2. Consider the cyclic permutation $\sigma = (1, 2, 3, 4, 5) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{smallmatrix}\right) \in S_5$ and let $H = \langle \sigma \rangle$ be the cyclic subgroup generated by σ.

(i) Show that H is a subgroup of A_5.

(ii) What is the index of H in A_5?

(iii) Is H a normal subgroup of A_5?

3. Let G be a group, let $H \leq G$ be a subgroup, and let $N \triangleleft G$ be a normal subgroup.

(i) Show that $H \cap N$ is a normal subgroup of H.

(ii) Suppose now that H is a normal subgroup of G. Show that $H \cap N$ is a normal subgroup of G.

4. Let G be set of all 2×2 matrices in $\mathrm{GL}_2(\mathbb{Z}_5)$ of the form $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$, with $c, d \in \mathbb{Z}_5$ and $d \neq 0$.

(i) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$.

(ii) Find the order of G.

(iii) Is G a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$?

(iv) Let N be the subset of all matrices in G of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}_5$. Show that N is a normal subgroup of G.

(v) Show that the factor group G/N is cyclic of order 4.

5. Let $G = \mathbb{Z}_{16} \times \mathbb{Z}_4$.

(i) Construct a surjective homomorphism $\varphi\colon G \to \mathbb{Z}_8$.

(ii) What is $\ker(\varphi)$?

(iii) Show that there is no surjective homomorphism $\varphi\colon G \to \mathbb{Z}_8 \times \mathbb{Z}_8$.