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MATH 3175 **Group Theory** **Summer 2, 2022**

Solutions for Assignment 4

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1. Let S_n be the symmetric group on n elements, let $A_n \le S_n$ be the alternating subgroup, consisting of all even permutations). Given a subgroup $H \le S_n$, show that either
(a) every permutation in H is even; or
(b) the set $H \cap A_n$ is properly contained in H, and, moreover, half the permutations in H are even and half are odd.

Suppose $H \not\subseteq A_n$. There is then an odd permutation $\sigma \in H$, and so $\sigma \in H \setminus H \cap A_n$, showing that $H \cap A_n \subsetneqq H$.

Now note that the map $f \colon S_n \to S_n$, $f(x) = \sigma x$ restricts to a bijection from A_n to its (left) coset, $\sigma A_n = S_n \setminus A_n$. Since $\sigma \in H$, this map further restricts to a bijection from $H \cap A_n$ to $\sigma(H \cap A_n)$, which is a bijection between the set of even permutations in H and the set of odd permutations in H. This proves the claim.

2. Consider the cyclic permutation $\sigma = (1,2,3,4,5) = \left(\begin{smallmatrix} 1&2&3&4&5\\ 2&3&4&5&1\end{smallmatrix}\right) \in S_5$ and let $H = \langle \sigma \rangle$ be the cyclic subgroup generated by σ.

(i) Show that H is a subgroup of A_5.

We have $(1,2,3,4,5) = (1,2)\cdot(2,3)\cdot(3,4)\cdot(4,5)$, and so $\sigma \in A_5$, which implies that $H = \langle\sigma\rangle$ is contained in A_5, and thus $H \le A_5$.

(ii) What is the index of H in A_5?

$[A_5 : H] = |A_5| \,/\, |H| = 60/5 = 12$.

(iii) Is H a normal subgroup of A_5?

Take $a = (1,2) \in S_5$. Then $a\sigma a^{-1} = (2,3)\cdot(3,4)\cdot(4,5)\cdot(1,2) = (1,3,4,5,2) \notin H$, and so H is not a normal subgroup of A_5. (Alternate proof: H is a non-trivial, proper subgroup of A_5; since A_5 is known to be a simple group, $H \ntriangleleft A_5$.)

3. Let G be a group, let $H \le G$ be a subgroup, and let $N \triangleleft G$ be a normal subgroup.

(i) Show that $H \cap N$ is a normal subgroup of H.

We already know that the intersection of two subgroups is a subgroup; thus, $H \cap N \le H$. Now let $g \in H \cap N$ and $h \in H$. Then $hgh^{-1} \in H$ (since H is a group) and $hgh^{-1} \in N$ (since N is a normal subgroup of G). Hence, $hgh^{-1} \in H \cap N$, showing that $H \cap N \triangleleft H$.

(ii) Suppose now that H is a normal subgroup of G. Show that $H \cap N$ is a normal subgroup of G.

Let $g \in H \cap N$ and $x \in G$. Then $xgx^{-1} \in H$ (since $H \triangleleft G$) and $xgx^{-1} \in N$ (since $N \triangleleft G$). Hence, $xgx^{-1} \in H \cap N$, showing that $H \cap N \triangleleft G$.

4. Let G be set of all 2×2 matrices in $\mathrm{GL}_2(\mathbb{Z}_5)$ of the form $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$, with $c, d \in \mathbb{Z}_5$ and $d \ne 0$.

(i) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$.

Let $A_1 = \begin{pmatrix} 1 & 0 \\ c_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ c_2 & d_2 \end{pmatrix}$ be two matrices in G. Then

$$A_1 \cdot A_2 = \begin{pmatrix} 1 & 0 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_1 + c_2 d_1 & d_1 d_2 \end{pmatrix}.$$

also belongs to G. Since $\mathrm{GL}_2(\mathbb{Z}_5)$ is a finite group, this shows that $G \le \mathrm{GL}_2(\mathbb{Z}_5)$.

(ii) Find the order of G.

Note that $c \in \mathbb{Z}_5$ and $d \in \mathbb{Z}_5^\times$, with no other constraints. Hence, $|G| = |\mathbb{Z}_5| \cdot |\mathbb{Z}_5^\times| = 5 \cdot 4 = 20$.

(iii) Is G a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$?

No, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \notin G$.

(iv) Let N be the subset of all matrices in G of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}_5$. Show that N is a normal subgroup of G.

Yes, since $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ a+bc & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -a/b & 1/b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ bc & 1 \end{pmatrix} \in N$.

(v) Show that the factor group G/N is cyclic of order 4.

Define a map $\varphi\colon G \to \mathbb{Z}_5^\times$ by sending a matrix $A = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$ to d. By the computation in part (i), we have that $\varphi(A_1A_2) = d_1d_2 = \varphi(A_1)\varphi(A_2)$, and this shows that φ is a homomorphism. Clearly, φ is surjective and its kernel is N. Thus, by the First Isomorphism Theorem, $G/N \cong \mathrm{im}(\varphi) = \mathbb{Z}_5^\times \cong \mathbb{Z}_4$, a cyclic group of order 4.

5. Let $G = \mathbb{Z}_{16} \times \mathbb{Z}_4$.

(i) Construct a surjective homomorphism $\varphi\colon G \to \mathbb{Z}_8$.

Set, for instance, $\varphi([x]_{16}, [y]_4) = [x]_8$.

(ii) What is $\ker(\varphi)$?

$\ker(\varphi) = \{0, [8]_{16}\} \times \mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

(iii) Show that there is no surjective homomorphism $\varphi\colon G \to \mathbb{Z}_8 \times \mathbb{Z}_8$.

Suppose $\varphi\colon G \to \mathbb{Z}_8 \times \mathbb{Z}_8$ is a surjective homomorphism. Let $K = \ker(\varphi)$. Then, by the First Isomorphism Theorem, $G/K \cong \mathrm{im}(\varphi) = \mathbb{Z}_8 \times \mathbb{Z}_8$, and so $|K| = |G| \,/\, |\mathbb{Z}_8 \times \mathbb{Z}_8| = 64/64 = 1$. This shows that K is the trivial subgroups, that is, φ is injective. Therefore, φ is an isomorphism.

But $([1]_{16}, [0]_4)$ is an element of order 16 in G, whereas every element in $\mathbb{Z}_8 \times \mathbb{Z}_8$ has order at most 8. Thus, $G \not\cong \mathbb{Z}_8 \times \mathbb{Z}_8$, which is a contradiction. Hence, the claim is proved.