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MATH 3175 **Group Theory** **Summer 2, 2022**

Solutions for Assignement 1

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1. Consider the binary operations $*$ and $\star$ on the set $S = \{e, a, b, c, d\}$ given by the following multiplication tables:

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	d	e	c
b	b	d	c	a	e
c	c	e	a	d	b
d	d	c	e	b	a

$\star$	e	a	b	c	d
e	e	a	b	c	d
a	a	c	e	d	b
b	b	d	c	a	e
c	c	e	d	b	a
d	d	b	a	e	c

Which (if either) of these binary operations gives S the structure of a group? Prove your answer.

The first table corresponds to the Cayley table of the (additive) cyclic group $\mathbb{Z}_5$, under the bijection $\{e, a, b, c, d\} \leftrightarrow \{0, 1, 2, 4, 3\}$. The second table is a Latin square that does not correspond to any group, since the $\star$ operation is not associative; for instance, $(a \star b) \star b = e \star b = b$, whereas $a \star (b \star b) = a \star c = d$.

2. Let G a group.

(i) Suppose $(ab)^{-1} = a^{-1}b^{-1}$, for all a and b in G. Prove that G is abelian.

We have:

$$(ab)^{-1} = a^{-1}b^{-1} \iff ((ab)^{-1})^{-1} = (a^{-1}b^{-1})^{-1} \iff ab = (b^{-1})^{-1}(a^{-1})^{-1} \iff ab = ba.$$

Thus, the hypothesis is equivalent to ($ab = ba$, $\forall a, b \in G$), which means that G must be abelian.

(ii) Give an example of a group G and two elements $a, b \in G$ for which $(ab)^{-1} \neq a^{-1}b^{-1}$.

Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Then $ij = k$ but $ji = -k$, and so $ij \neq ji$, or, $(ij)^{-1} \neq j^{-1}i^{-1}$.

3. Let G and H be two groups, and let $G \times H$ be their product.

(i) If both G and H are commutative, show that $G \times H$ is also commutative.

Let (g_1, h_1) and (g_2, h_2) two arbitrary elements in $G \times H$. Then:

$$\begin{aligned}
(g_1, h_1)(g_2, h_2) &= (g_1g_2, h_1h_2) && \text{by the definition of the group operation in } G \times H \\
&= (g_2g_1, h_2h_1) && \text{since both } G \text{ and } H \text{ are commutative} \\
&= (g_2, h_2)(g_1, h_1) && \text{by the definition of the group operation in } G \times H
\end{aligned}$$

(ii) If either G or H is non-commutative, show that $G \times H$ is non-commutative.

Suppose G is non-commutative. Then there are $g_1, g_2 \in G$ such that $g_1g_2 \neq g_2g_1$. Using the same rationale as above, we infer that $(g_1, h_1)(g_2, h_2) \neq (g_2, h_2)(g_1, h_1)$, for any $h_1, h_2 \in H$ (for instance, we could take $h_1 = h_2 = e_H$ to be concrete), and this shows that $G \times H$ is non-commutative.

The case when H is non-commutative is treated completely similarly.

4. Let G be a group, with group operation $\cdot$ and identity $e = 1$. Let u be an element not in G and consider the magma

$$M = G \cup (Gu),$$

where $Gu = \{gu \mid g \in G\}$ and the product in M is given by the usual product of elements in G, together with $1 \cdot u = u$ and

$$(gu)h = (gh^{-1})u \tag{1}$$

$$g(hu) = (hg)u \tag{2}$$

$$(gu)(hu) = h^{-1}g. \tag{3}$$

(i) Show that $u^2 = 1$ and $ug = g^{-1}u$.

We have: $u^2 = (1 \cdot u)(1 \cdot u)$, which by rule (3) is equal to $1^{-1}1 = 1$. Likewise, $ug = (1 \cdot u)g$, which, by rule (1) [with $g \to 1, h \to g$] is equal to $(1 \cdot g^{-1})u = g^{-1}u$.

(ii) Show that M has an identity.

Since $e = 1$ is the identity of G, we gave $1 \cdot g = g \cdot 1 = g$ for all $g \in G$. Moreover, $1 \cdot (gu) = (1 \cdot g)u = gu$ by rule (2) [with $g \to 1, h \to g$] and $(gu) \cdot 1 = (g \cdot 1^{-1})u = gu$ by rule (1) [with $h \to 1$]. Thus, 1 is an identity for M.

(iii) Show that the multiplication on M is associative if and only if G is abelian.

First assume G is abelian, that is, $gh = hg$ for all $h, g \in G$. Then:

$g(hu) = (hg)u = (gh)u$ by (2) and commutativity of G

$(gu)h = (gh^{-1})u = g(h^{-1}u) = g(uh)$ by (1), previous line [with $h \to h^{-1}$], and part (i) [with $g \to h$]

Conversely, assume M is associative, and let $g, h \in G$. Using part (i), the associativity hypothesis, and formula (2), we find:

$$gh = ghu^2 = g(hu)u = (hg)u^2 = hg.$$

5. Consider the set of matrices $S = \{I, A, B, C\}$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Write out the multiplication table for S.

I	A	B	C
A	I	C	B
B	C	I	A
C	B	A	I

(ii) Show that the set S (with this multiplication) is a magma. Is this magma abelian?

S is closed under this operation, and thus it is a magma. Clearly, I is an identity for S, and $AB = BA$, $AC = CA$, and $BC = CB$; thus, S is abelian.

(iii) Is the magma S a group? Every element in S has an inverse (namely, itself), and matrix multiplication is associative. Thus, S is a group.

6. Give an example of three permutations $\alpha, \beta, \gamma \in S_4$ (none of which is equal to the identity permutation) such that $\alpha\beta = \beta\alpha$ and $\beta\gamma = \gamma\beta$ but $\alpha\gamma \neq \gamma\alpha$.

Take for instance

$$\begin{aligned}\alpha &= (12)(34) = \left(\begin{smallmatrix}1&2&3&4\\2&1&4&3\end{smallmatrix}\right)\\ \beta &= (14)(23) = \left(\begin{smallmatrix}1&2&3&4\\4&3&2&1\end{smallmatrix}\right)\\ \gamma &= (14) = \left(\begin{smallmatrix}1&2&3&4\\4&2&3&1\end{smallmatrix}\right).\end{aligned}$$

It is readily verified that $\alpha\beta = \beta\alpha = (13)(24)$ and $\beta\gamma = \gamma\beta = (24)$ yet $\alpha\gamma = (1243)$, which differs from $\gamma\alpha = (1342)$.