Name: Solutions

Prof. Alexandru Suciu

MATH 3150 Real Analysis Spring 2011

# Midterm Exam

**Instructions**: Write your name in the space provided. Calculators are permitted, but no notes are allowed. Each problem is worth 10 points .

---

**1.** Let $a_1 = 0$, $a_2 = 1$, $a_3 = \sqrt{3}$, ..., $a_n = \sqrt{1 + 2a_{n-1}}$.

(a) Show that the sequence $\{a_n\}$ is strictly increasing.

Induction on $n$:

- $a_2 = 1 > 0 = a_1$ ✓
- Assume: $a_n > a_{n-1}$
- Then: $a_{n+1} = \sqrt{1+2a_n} > \sqrt{1+2a_{n-1}} = a_n$

(b) Show that the sequence $\{a_n\}$ is bounded above.

Show by induction on $n$ that $a_n < 3$:

- $a_1 = 0 < 3$ ✓
- Assume: $a_n < 3$
- Then: $a_{n+1} = \sqrt{1+2a_n} < \sqrt{1+2\cdot 3} < \sqrt{9} = 3$

(c) Show that the sequence $\{a_n\}$ is converging. Give a reason for your answer.

$\{a_n\}$ is monotone increasing and bounded above.

Hence $\{a_n\}$ is converging.

(d) Find $\lim_{n\to\infty} a_n$. Set $a := \lim_{n\to\infty} a_n$. Then:

$a = \sqrt{1+2a} \to a^2 = 1+2a \to a^2 - 2a - 1 = 0$
$\to a = 1 \pm \sqrt{2}$

But $a_n > 0 \Rightarrow a > 0 \Rightarrow a = 1 + \sqrt{2}$

**2.** Let $\{x_n\}$ be a sequence in a complete metric $(X, d)$.

(a) Suppose $d(x_{n+1}, x_n) \leq 1/2^n$, for all $n \geq 1$. Show that $\{x_n\}$ converges.

Since $X$ is complete, it is enough to show that $\{x_n\}$ is a Cauchy sequence.

We have, for all $p \geq 1$:

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_n)$$

(by the triangle inequality)

$$\leq 1/2^n + \cdots + \frac{1}{2^{n+p-1}}$$ (by assumption)

$$= \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^p}\right)$$

$$\leq \frac{1}{2^{n-1}}$$

But $\frac{1}{2^{n-1}} \longrightarrow 0$, so $\forall \varepsilon > 0$ $\exists N$ s.t. for $n \geq N$:

$d(x_n, x_{n+p}) < \varepsilon$, for all $p \geq 1$, i.e., $\{x_n\}$ is Cauchy.

(b) Suppose $d(x_{n+1}, x_n) \leq 1/n$, for all $n \geq 1$. Show by example that $\{x_n\}$ may *not* converge.

Take $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$ $(n \geq 2)$

Then:

$$d(x_{n+1}, x_n) = |x_{n+1} - x_n| = \frac{1}{n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (eg., by integral test)

the sequence of partial sums diverges, i.e, $\{x_n\}$ does <u>not</u> converge.

**3.** Let $A$ and $B$ be two non-empty subsets of $\mathbb{R}$, and write $A+B = \{x+y \mid x \in A \text{ and } y \in B\}$.

(a) Show that
$$\sup(A+B) = \sup(A) + \sup(B).$$

Set $a := \sup(A)$, $b := \sup(B)$.

($\leq$) $x \in A, y \in B \Rightarrow x \leq a, y \leq b \Rightarrow x+y \leq a+b$

$\therefore \sup(A+B) \leq \sup(A) + \sup(B)$

($\geq$) Let $\varepsilon > 0$. Then:

$\exists\, x \in A$ s.t. $a - \varepsilon/2 \leq x$

$\exists\, y \in B$ s.t. $b - \varepsilon/2 \leq y$ (by definition of sup)

$\therefore a+b-\varepsilon \leq x+y$

$\therefore a+b-\varepsilon \leq \sup(A+B)$

Since this is true for all $\varepsilon > 0$, we get

$\sup(A) + \sup(B) \leq \sup(A+B)$

(b) Now suppose $A = \{x \in \mathbb{R} \mid x^2 < 2\}$ and $B = \{y \in \mathbb{R} \mid y < 3\}$. Compute:
$$\sup(A), \quad \sup(B), \quad \sup(A+B).$$

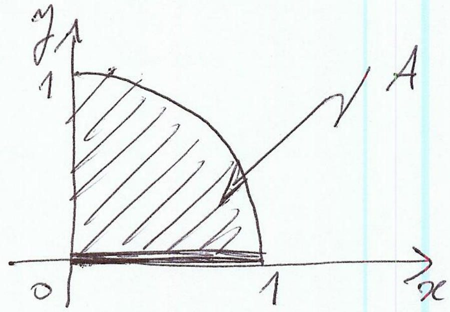
$\sup(A) = \sqrt{2}$

$\sup(B) = 3$

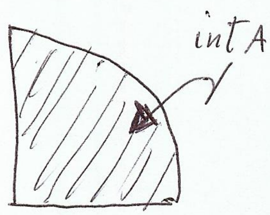
$\sup(A+B) = \sqrt{2} + 3$

**4.** Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ x \geq 0,\ y > 0\}$.

(a) Draw a picture of the set $A$.



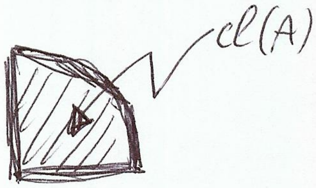

(b) What is the interior of $A$? Is $A$ an open subset of $\mathbb{R}^2$?




$$\mathrm{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1,\ x>0,\ y>0\}$$

$$\mathrm{int}(A) \neq A \Rightarrow A \text{ \underline{not} open}$$

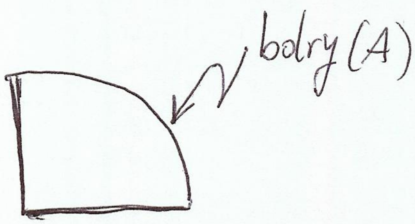
(c) What is the closure of $A$? Is $A$ a closed subset of $\mathbb{R}^2$?




$$\mathrm{cl}(A) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$$

$$\mathrm{cl}(A) \neq A \Rightarrow A \text{ \underline{not} closed}$$

(d) What is the boundary of $A$?




$$\mathrm{bdry}(A) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1,\ x \geq 0,\ y \geq 0\}$$
$$\cup \{(x,0) \mid 0 \leq x \leq 1\}$$
$$\cup \{(0,y) \mid 0 \leq y \leq 1\}$$

**5.** Let $(X, d)$ be a metric space, $A$ a subset of $X$, and $x$ a point in $X$. We say that:

- The point $x$ is an *accumulation* point for $A$ if every open set $U$ containing $x$ contains some point of $A$ other than $x$.
- The point $x$ is a *limit* point for $A$ if every open set $U$ containing $x$ contains some point of $A$.

(a) Suppose $x$ is a limit point for $A$, and $x \notin A$; then show that $x$ is an accumulation point for $A$.

Let $U$ be an open set, $U \ni x$

Since $x$ is a limit point for $A$:

$\exists y \in A \cap U$.

Since $x \notin A$ : $y \neq x$.

Hence, $x$ is an accumulation point for $A$.

(b) Let $A = \{1/n \mid n \in \mathbb{N}\}$, viewed as a subset of $\mathbb{R}$.

- What are the limit points of $A$?
- What are the accumulation points of $A$?
- Does the set of limit points coincide with the set of accumulation points?

- Limit points: $A \cup \{0\}$
- Accumulation points: $\{0\}$
- No!

**6.** Decide whether each of the following series converges or not. In each case, indicate which test is used, and why that test yields the conclusion you are drawing.

(a)

$$\sum_{k=1}^{\infty} \frac{100^k}{k!}$$

(Limit) Ratio test:

$$\frac{\left|\frac{100^{k+1}}{(k+1)!}\right|}{\left|\frac{100^k}{k!}\right|} = \frac{100}{k+1} \xrightarrow[k\to\infty]{} 0 < 1$$

$\therefore$ Series converges

(b)

$$\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^4 - 1}}$$

Comparison test:

$$\frac{k}{\sqrt{k^4-1}} \geq \frac{k}{\sqrt{k^4}} = \frac{k}{k^2} = \frac{1}{k}$$

$\sum_{k=2}^{\infty} \frac{1}{k}$ diverges (say by integral test)

$\therefore$ given series diverges.