Name: Solutions

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NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 3150 **Real Analysis** **Spring 2011**

Final Exam

1. Let $f_1, \ldots, f_k$ be functions defined on a subset $A \subseteq \mathbb{R}^n$ and taking values in $\mathbb{R}$. Let $f = \sum_{i=1}^{k} f_i$, and set

$$m_i = \inf\{f_i(x) \mid x \in A\}, \qquad m = \inf\{f(x) \mid x \in A\},$$

(a) Show that $m \geq \sum_{i=1}^{k} m_i$.

$f_i(x) \geq m_i$ for $i = 1, \ldots, k$

$\therefore\ f(x) = \sum_{i=1}^{k} f_i(x) \geq \sum_{i=1}^{k} m_i$

$\therefore\ m = \inf_{x \in A} \{f(x)\} \geq \sum_{i=1}^{k} m_i$

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(b) Given an example where equality fails.

$f_1, f_2 : [0,1] \longrightarrow \mathbb{R}$

$f_1(x) = x \longrightarrow m_1 = 0$

$f_2(x) = -x \longrightarrow m_2 = -1$

$f = f_1 + f_2 = 0 \longrightarrow m = 0$

$m_1 + m_2 = -1 < 0 = m$

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2. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be two subsets, and consider their product, $A \times B$, viewed as a subset in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

(a) Suppose A and B are path-connected. Show that $A \times B$ is path-connected.

Let (a_1, b_1) & (a_2, b_2) be in $A \times B$.

1 A path-conn $\Rightarrow \exists$ path $\gamma_1 : [0,1] \longrightarrow A$
$\gamma_1(0) = a_1, \gamma_2(1) = a_2$

1 B path-conn $\Rightarrow \exists$ pat $\gamma_2 : [0,1] \longrightarrow B$
$\gamma_2(0) = b_1, \gamma_2(1) = b_2$

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3 $\therefore$ $\gamma : [0,1] \longrightarrow A \times B$
$\gamma(t) = (\gamma_1(t), \gamma_2(t))$
is path in $A \times B$ from (a_1, b_1) to (a_2, b_2)

(b) Suppose A and B are bounded. Show that $A \times B$ is bounded.

1 A bounded $\Rightarrow A \subset B(0, r)$ for some $r > 0$
1 B bounded $\Rightarrow B \subset B(0, s)$ for some $s > 0$

3 $\therefore$ $A \times B \subset B(0, r) \times B(0, s)$
$\subset B(0, \sqrt{r^2 + s^2})$

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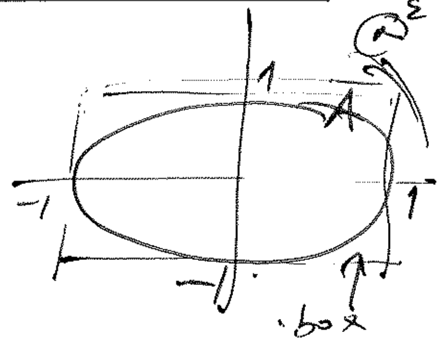
3. Consider the following subset of the plane:

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^4 = 1\}.$$

Show that A is compact.



• A closed: $A = f^{-1}(\{1\})$

where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = x^2 + y^4$ is cont.

Since $\{1\}$ closed $\Longrightarrow$ A closed (4)

Alternate sol: $\forall (x,y) \in A^c$, $\exists\, \varepsilon > 0$ s.t. $B((x,y), \varepsilon) \subset A^c$

$\therefore$ A^c open $\therefore$ A closed.

• A bounded

If either $|x| > 1$ or $|y| > 1$, then $(x,y) \notin A$

$\therefore\ A \subset \underbrace{D(0,1) \times D(0,1)}_{\text{box}} \subset \underbrace{D(0,\sqrt{2})}_{\text{disk}}$ (4)

By Heine-Borel, A is compact (2)

4. Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1 - \frac{1}{n}, \text{ for some integer } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Riemann-integrable, and that $\int_0^1 f(x)dx = 0$.

$f \geq 0 \implies \underline{\int_a^b} f(x)dx \geq 0$. 2

So enough to find a sequence of partitions P_n such that

$$U(f, P_n) \xrightarrow[n\to\infty]{} 0$$ 1

Let P_n be the partition of $[0,1]$ obtained by subdividing each interval $[1-\frac{1}{i}, 1-\frac{1}{i+1}]$ into n equal intervals, for $i = 1, \ldots, n$

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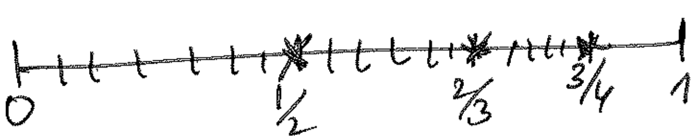

Then

$$U(f,P_n) = \sum_{i=1}^{n} \frac{2}{n}\left[\left(1-\frac{1}{i+1}\right) - \left(1-\frac{1}{i}\right)\right] + \left(1-1+\frac{1}{n+1}\right)$$

$$= \frac{2}{n}\left(-\frac{1}{n+1}\right) + \frac{1}{n+1} = \frac{n-2}{n(n+1)} \xrightarrow[n\to\infty]{} 0$$

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5. Consider the function $f\colon [0,\pi] \to \mathbb{R}$ given by

$$f(x) = \int_0^{x^2} \cos\left(\sqrt{t}\,\right) dt$$

(a) What is $f(0)$?

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$$f(0) = \int_0^0 \cos(\sqrt{t})\, dt = 0$$

(b) Show that f is differentiable. What is its derivative?

$h(t) = \cos\sqrt{t}$ is cont on $[0,\pi]$

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$\therefore$ $g(x) = \int_0^x \cos(\sqrt{t})\, dt$ is differentiable, by FTC

& $g'(x) = \cos(\sqrt{x})$

$\therefore$ $f(x) = g(x^2)$ is differentiable, by Chain Rule

and $f'(x) = g'(x^2) \cdot 2x = \cos(\sqrt{t}\,|_{t=x^2}) \cdot 2x$

$= \cos(x) \cdot 2x$

(c) When $x = \pi/3$, show that $f'(x) = x$.

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$$f'(\pi/3) = \cos(\pi/3) \cdot 2 \cdot \pi/3$$
$$= \frac{1}{2} \cdot 2 \cdot \pi/3 = \pi/3$$

6. Let $f\colon [0,+\infty) \to \mathbb{R}$ be a continuous function, differentiable on $(0,\infty)$. Suppose that

$$f(x) + x \cdot f'(x) \geq 0, \quad \text{for all } x > 0.$$

Show that $f(x) \geq 0$, for all $x \geq 0$.

3 Let $g : [0,\infty) \longrightarrow \mathbb{R}$ $\quad g(x) = x\, f(x)$.

Then $g'(x) = f(x) + x f'(x)$ by Product Rule

2 $\quad \geq 0$ by assumption for all $x > 0$

$\therefore$ 2 g increasing for all $x > 0$

~~$\therefore$ f increasing for all $x > 0$ (since $g = x f(x)$)~~

1 But $g(0) = 0 \Rightarrow g(x) \geq 0$ for all $x \geq 0$

1 $\therefore$ $f(x) = \dfrac{g(x)}{x} \geq 0$ for $x > 0$

1 Then also $f(0) = \lim_{x \to 0} f(x) \geq 0$ since f cont at 0

$\therefore$ $f(x) \geq 0$, $\forall x \geq 0$

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $u, v, w\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions, and let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be the function given by $F(x,y,z) = f(u(x,z), v(x,y), w(y,z))$.

(a) Use the Chain Rule to express $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$ in terms of the partial derivatives of f, u, v, and w.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial z}\underbrace{\cancel{\frac{\partial w}{\partial x}}}_{0}$$

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$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}\underbrace{\cancel{\frac{\partial u}{\partial y}}}_{0} + \frac{\partial f}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial w}{\partial y}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial y}\underbrace{\cancel{\frac{\partial v}{\partial z}}}_{0} + \frac{\partial f}{\partial z}\frac{\partial w}{\partial z}$$

(b) Now suppose

$$f(x,y,z) = x^3 - x^2y^2 + z^4, \quad u(x,y) = x + y, \quad v(x,y) = 2xy, \quad w(x,y) = x^2 + y^3.$$

Compute $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$, either using part (a), or directly (or both).

$$F(x,y,z) = (x+z)^3 - (x+z)^2(2xy)^2 + (y^2+z^3)^4$$

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$$\partial F/\partial x = 3(x+z)^2 - 8(x+z)x^2y^2 - 8xy^2(x+z)^2$$

$$\partial F/\partial y = -8x^2y(x+z)^2 + 8y(y^2+z^3)^3$$

$$\partial F/\partial z = 3(x+z)^2 - 8x^2y^2(x+z) + 12z^2(y^2+z^3)^3$$

8. Consider the surface S in $\mathbb{R}^3$ given by the equation $x^2 + yz - xz = 4$.

(a) Find a unit normal vector to S at the point $(1, 2, 3)$.

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Set $f(x,y,z) = x^2 + yz - xz$

Then $f(1,2,3) = 1^2 + 2\cdot 3 - 1\cdot 3 = 1 + 6 - 3 = 4$

so $S = \{f = 4\}$ and $v \in S$

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3 $\nabla f = (2x - z, z, y - x)$

$\nabla f(1,2,3) = (-1, 3, 1)$

$\therefore$ normal vector to S at $(1,2,3)$ is

1 $\vec{u} = \dfrac{\nabla f(1,2,3)}{\|\nabla f(1,2,3)\|} = \dfrac{(-1,3,1)}{\sqrt{11}}$

(b) Find the equation of the tangent plane to the surface S at the point $(1, 2, 3)$.

$-(x-1) + 3(y-2) + (z-3) = 0$

$\therefore \boxed{-x + 3y + z = 8}$

$z = x - 3y + 8$

$x - 3y - z = -8$