Professor Alex Suciu

MATH 3150 **Real Analysis** **Fall 2022**

A solution to Problem 9.15

Problem. Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$, for all $a \in \mathbb{R}$.

Solution. If $a = 0$, this is obvious. It is enough to prove the claim when $a > 0$, since, in general, $\lim_{n\to\infty} |x_n| = 0$ implies $\lim_{n\to\infty} x_n = 0$.

So fix $a > 0$. By the Archimedean principle, there is an integer $n > a$. Set

$$k := \frac{a^n}{n!}.$$

Then, for every $m > n$, we have

$$0 < \frac{a^m}{m!} = k \cdot \frac{a}{n+1} \cdots \frac{a}{m-1}\frac{a}{m} < k \cdot 1 \cdots 1 \cdot \frac{a}{m} = k\frac{a}{m}.$$

But

$$\lim_{m\to\infty} k\frac{a}{m} = ka \lim_{m\to\infty} \frac{1}{m} = ka \cdot 0 = 0.$$

Thus, by the "Squeeze Theorem" (from Assignment 2, but slightly modified, to allow for the "squeezing inequalities" to hold only for m sufficiently large), we conclude that $a^m/m! \to 0$.

This can also be seen directly from the definition of limit: let $\epsilon > 0$, and take $N > ka/\epsilon$. Then, for $m > N$, we have

$$|a^m/m!| < ka/m < ka/N < \epsilon,$$

thus showing once again that $a^m/m! \to 0$.