Name: ____________________

MATH 3150 **Real Analysis: Solutions to the Midterm** **Fall 2022**

Instructions: Write your name in the space provided. Calculators are permitted. You are also allowed a one-sided A4 sized note sheet of only definitions and theorems (no examples allowed) from classes and the textbook. Make sure your name is on the sheet, and hand in your note sheet along with your exam. Books, other notes, and laptops are **not** allowed.

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1. Define a sequence (x_n) in $\mathbb{R}$ recursively by setting $x_1 = 7$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$.

(a) (10 pts) Show that the sequence converges.

Solution: The first step is to see by induction that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. For the base case, we have

$$x_2 = \sqrt{2 + x_1} = \sqrt{2 + 7} = \sqrt{9} = 3$$

so $3 = x_2 < x_1 = 7$. For the inductive step, we assume $x_{n+1} < x_n$ and show that $x_{n+2} < x_{n+1}$.

$$\begin{aligned} x_{n+1} &< x_n \quad \text{so} \\ 2 + x_{n+1} &< 2 + x_n \quad \text{so} \\ \sqrt{2 + x_{n+1}} &< \sqrt{2 + x_n} \quad \text{and we have} \\ x_{n+2} &< x_{n+1} \end{aligned}$$

This completes the proof that the sequence (x_n) is decreasing. The next step is to see that (x_n) is bounded below. The proof is by induction. Note that the base case is $x_1 > 0$ which holds since $x_1 = 7$. For the inductive step, assume $x_n > 0$ and show that $x_{n+1} > 0$. Since $x_n > 0$ we have that $2 + x_n > 0$. Hence, $\sqrt{2 + x_n} = x_{n+1} > 0$ and it follows that the sequence (x_n) is bounded below by 0.

Since (x_n) is decreasing and bounded below, the sequence converges.

(b) (10 pts) Find the limit of the sequence.

Solution: Let $\lim x_n = L$, then $L = \lim x_{n+1}$ and then using theorems in the text about limits of sequences we have

$$\begin{aligned} L &= \lim x_{n+1} \\ &= \lim \sqrt{2 + x_n} \\ &= \sqrt{\lim(2 + x_n)} \\ &= \sqrt{2 + \lim x_n} \\ &= \sqrt{2 + L}. \end{aligned}$$

Since $L = \sqrt{2 + L}$ we have $L^2 = 2 + L$ so $L^2 - L - 2 = (L + 1)(L - 2) = 0$, and hence, $L = -1$ or $L = 2$. Since the sequence is bounded below by 0, it follows that $L \neq -1$, so $\lim x_n = L = 2$.

2. (15 pts) Let $a_n = (-1)^n + 1/n$. Use the definition of lim inf and lim sup to find $\limsup a_n$ and $\liminf a_n$.

Solution: Set

$$v_N = \sup\{a_n \colon n \in \mathbb{N}, n \geq N\}$$
$$u_N = \inf\{a_n \colon n \in \mathbb{N}, n \geq N\}$$

then $\limsup a_n = \lim v_N$ and $\liminf a_n = \lim u_n$.

Note that

$$\begin{aligned}
&a_{2k} = 1 + 1/2k,\\
&a_{2k+1} = -1 + 1/(2k+1),\\
&a_{2k} > a_{2k+1},\\
&\lim_{k\to\infty} a_{2k} = 1, \quad \text{and}\\
&\lim_{k\to\infty} a_{2k+1} = -1.
\end{aligned}$$

Since $a_{2k} > a_{2k+1}$, it follows that $v_N = \sup\{a_{2k} : 2k \geq N\}$ which equals $1 + 1/N$ if N is even and $1 + 1/(N+1)$ if N is odd. Hence $\limsup a_n = \lim_{k\to\infty} a_{2k} = 1$.

Since $a_{2k} > a_{2k+1}$, it follows that $u_N = \inf\{a_{2k+1} \colon 2k+1 \geq N\}$ which equals $\lim_{k\to\infty}[-1 + 1/(2k+1)] = -1$. Thus, $u_N = -1$ for all N and we have $\liminf a_n = -1$.

3. (10 pts) Show that a subsequence of a subsequence of a sequence is a subsequence of the original sequence.

Solution: Recall that a map $f\colon \mathbb{N} \to \mathbb{N}$ is called increasing if $f(n) > f(m)$ for all $n > m$.

A first step is to see that a composition $f_2 \circ f_1$ of increasing functions is an increasing function. Suppose f_1 and f_2 are increasing functions and $n > m$, then $f_1(n) > f_1(m)$ since f_1 is increasing and then $f_2(f_1(n)) > f_2(f_1(m))$ since f_2 is increasing and $f_1(n) > f_1(m)$ and the argument that a composition of increasing functions is increasing is complete.

Recall that a sequence is a function $g\colon \mathbb{N} \to \mathbb{R}$ and a subsequence of g is a composition $f_1 \circ g$ with $f_1\colon \mathbb{N} \to \mathbb{N}$ an increasing function. Thus, a subsequence of the subsequence $f_1 \circ g$ is a composition $f_2\big(f_1 \circ g\big)$ with $f_2\colon \mathbb{N} \to \mathbb{N}$ an increasing function. Note that $f_2\big(f_1 \circ g\big) = \big(f_2 \circ f_1\big) \circ g$, with $f_2 \circ f_1$ increasing since it is the composition of increasing functions.

Thus, the subsequence $f_2\big(f_1 \circ g\big)$ of the subsequence $f_1 \circ g$ of g is the subsequence $\big(f_1 \circ f_2\big) g$ of the sequence g.

4. (15 pts) Let (x_n) and (y_n) be Cauchy sequences of real numbers, and let $z_n = x_n - y_n$. Use the definition of Cauchy sequence to show that (z_n) is also a Cauchy sequence.

Solution: Given any $\epsilon > 0$, since (x_n) and (y_n) are Cauchy, it follows that there are elements $N_1, N_2 \in \mathbb{N}$ with

$$\begin{aligned} |x_n - x_m| &< \epsilon/2 \quad \text{for all } n, m > N_1 \\ |y_n - y_m| &< \epsilon/2 \quad \text{for all } n, m > N_2. \end{aligned}$$

Thus for $N = \max\{N_1, N_2\}$ we have for all $n, m > N$, that

$$\begin{aligned} |(x_n - y_n) - (x_m - y_m)| &= |x_n - x_m + y_m - y_n| \\ &\leq |x_n - x_m| + |y_m - y_n| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This completes the argument that $(x_n - y_n)$ is a Cauchy sequence.

5. Let $f\colon [1,2] \to \mathbb{R}$ be a function such that $f(1) > 1$ and $f(2) < 4$.

(a) (10 pts) Assuming f is continuous, show that there is a number $c \in [1,2]$ such that $f(c) = c^2$.

Solution: Consider the function $g\colon [1,2] \to \mathbb{R}$, $g(x) = x^2$. This is a continuous function; indeed, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} x_n^2 = x_0^2$, by properties of limits of sequences. Therefore, the function

$$h\colon [1,2] \to \mathbb{R}, \quad h(x) = f(x) - x^2$$

is also a continuous function, since $h = f - g$, and the difference of two continuous functions is again continuous. Furthermore, note that $h(1) = f(1) - 1^2 > 1 - 1 = 0$ and $h(2) = f(2) - 2^2 < 4 - 4 = 0$.

To sum up, the function $h\colon [1,2] \to \mathbb{R}$ is continuous, $h(1) > 0$, and $h(2) < 0$. Therefore, by the Intermediate Value Theorem, there is a number $c \in [1,2]$ such that $h(c) = 0$; that is, $f(c) - c^2 = 0$, or, $f(c) = c^2$.

(b) (10 pts) Give an example of a (discontinuous) function $f\colon [1,2] \to \mathbb{R}$ for which the equation $f(c) = c^2$ has no solution $c \in [1,2]$.

Solution: Consider the function $f\colon [1,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } 1 \le x < \sqrt{2} \\ 0 & \text{if } \sqrt{2} \le x \le 2. \end{cases}$$

Clearly, the function f satisfies the hypothesis of the problem; indeed, $f(1) = 2 > 1$ and $f(2) = 0 < 4$. Note also that f is not continuous at $x = \sqrt{2}$, since $\lim_{x\to\sqrt{2}^-} f(x) = 2$, while $\lim_{x\to\sqrt{2}^+} f(x) = 0$. Finally, note that

$$x^2 < 2 = f(x) \quad \text{if } 0 \le x < \sqrt{2}$$
$$0 = f(x) < x^2 \quad \text{if } \sqrt{2} \le x \le 2.$$

Therefore, $f(x) \ne x^2$ for $0 \le x \le 2$; that is, the equation $f(c) = c^2$ has no solution $c \in [1,2]$.

6. (10 pts) Let f be a function defined on a domain $D \subset \mathbb{R}$. Given elements $x, y \in D$ with $x \neq y$ set $s(x, y) = (f(x) - f(y))/(x - y)$ and then let

$$S = \{s(x, y) \colon x, y \in D, x \neq y\}$$

Show directly from the definition of uniform continuity that if the set S is bounded, then f is uniformly continuous on D.

Solution: In general, if a set S is bounded, then the set of absolute values, $|S| = \{|s| \colon s \in S\}$ is also bounded. To see this recall that a set S is bounded if it is a subset of a closed interval; that is, $L \leq s \leq U$ for all $s \in S$; equivalently $S \subset [L, U]$. Now consider the set $-S = \{-s \colon s \in S\}$, then $-S \subset [-U, -L]$ and $S \cup (-S) \subset [L, U] \cup [-U, -L] \subset \big[a = \min\{L, -U\}, b = \max\{U, -L\}\big]$. Since $|a| = a$ if $a > 0$ and $-a$ if $a < 0$, it follows that $|S| \subset S \cup (-S) \subset [a, b]$, and hence, $|S|$ is a bounded set.

Now taking S to be the set in this problem we have that $|S|$ is bounded. Hence, there is an $M > 0$ with

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M \quad \text{for all } x, y \in D, x \neq y \tag{1}$$

Given any $\epsilon > 0$, for all $x, y \in D$ with $x \neq y$ and $|x - y| < \epsilon/M$, we have

$$|f(x) - f(y)| < M|x - y| < M \cdot \epsilon/M = \epsilon.$$

Thus, f is uniformly continuous on D.

7. (10 pts) Let X, Y and Z be subsets of $\mathbb{R}$. Suppose $f\colon X \to Y$ is a uniformly continuous function on X, and $g\colon Y \to Z$ is a uniformly continuous function on Y. Show that the composition $g \circ f\colon X \to Z$ is also uniformly continuous.

Solution: Let $\epsilon > 0$. Since g is uniformly continuous on Y, there is a $\delta_1 > 0$ such that, for all $y_1, y_2 \in Y$ with

$$|y_1 - y_2| < \delta_1 \Longrightarrow |g(y_1) - g(y_2)| < \epsilon.$$

Likewise, since f is uniformly continuous on X, there is a $\delta > 0$ such that, for all $x_1, x_2 \in X$ with

$$|x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \delta_1.$$

Therefore, if $|x_1 - x_2| < \delta$, taking $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we have $|y_1 - y_2| < \delta_1$, and hence

$$|g(f(x_1)) - g(f(x_2))| = |g(y_1) - g(y_2)| < \epsilon.$$

To recap, we have shown the following: For all $\epsilon > 0$ and for all $x_1, x_2 \in X$ with $|x_1 - x_2| < \delta$, we have that $|(g \circ f)(x_1) - (g \circ f)(x_2)| < \epsilon$, and this means precisely that the function $g \circ f$ is uniformly continuous on X.