1. (10 pts) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{n^3} \quad \text{for all } n \in \mathbb{N}$$

Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

Solution: Let $n \geq m$. Then, by the triangle inequality and our assumption,

$$\begin{aligned} |s_n - s_m| &\leq |s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m| \\ &< \frac{1}{(n-1)^3} + \cdots + \frac{1}{m^3} \\ &= \sum_{k=m}^{n-1} \frac{1}{k^3}. \end{aligned}$$

Now consider the series $\sum_{k=1}^{\infty} 1/k^3$. By the integral test, this is a convergent series. Thus, by the Cauchy criterion for convergence, the following holds: for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that, for all $n \geq m \geq N$,

$$\sum_{k=m}^{n-1} \frac{1}{k^3} < \epsilon$$

Returning now to the original sequence, the above estimates imply the following: for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$, for all $n \geq m \geq N$. This shows that (s_n) is a Cauchy sequence (of real numbers), and thus, it converges.

2. Consider the sequence (x_n) with terms $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \ldots$.

   (a) (5 pts) Show that (x_n) is bounded.

   The terms of the sequence are fractions of the form p/q with $1 \leq p < q$. Hence, $0 < x_n < 1$, for all $n \geq 1$, showing that the sequence is bounded below by 0 and above by 1.

   (b) (10 pts) Show directly from the definition that (x_n) is *not* a Cauchy sequence.

   We need to show the following: There exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there are some $n \geq m \geq N$ so that $|x_n - x_m| \geq \epsilon$.

   Let us take $\epsilon = 1/2$. Then, for all $N \in \mathbb{N}$, there will be an $m \geq N$ such that $x_m = \frac{k}{k+1}$ for some $k \geq 3$. Then $x_{m+1} = \frac{1}{k+2}$, and so

$$|x_{m+1} - x_m| = \left| \frac{1}{k+2} - \frac{k}{k+1} \right| = \left| \frac{k+1-k^2-2k}{(k+1)(k+2)} \right| = \frac{k^2+k-1}{k^2+3k+2} \geq \frac{1}{2},$$

   where the last inequality holds, since $2k^2 + 2k - 2 \geq k^2 + 3k + 2$, or $k^2 - k - 4 \geq 0$, which is true for all $k \geq 3$.

   (c) (5 pts) Find two convergent subsequences of (x_n) that converge to two different limits.

   The subsequence $x_1 = 1/2$, $x_2 = 1/3$, $x_4 = 1/4$, $x_7 = 1/5$, $x_{11} = 1/6$, $x_{16} = 1/7, \ldots$ has terms $x_{k(k-1)/2+1} = 1/(k+1)$, and thus converges to 0.

   The subsequence $x_1 = 1/2$, $x_3 = 2/3$, $x_6 = 3/4$, $x_{10} = 4/5$, $x_{15} = 5/6, \ldots$ has terms $x_{k(k+1)/2} = k/(k+1)$, and thus converges to 1.

(d) (5 pts) What conclusion regarding the convergence of the sequence (x_n) can you draw from part (c), and how does that conclusion compare to the answer in part (b)?

Solution: The given sequence has two subsequences that converge to two distinct limits (0 and 1). Therefore, the sequence (x_n) is not convergent, since if we had $\lim_{n\to\infty} x_n = x$, then any subsequence would also converge to x, forcing $0 = x = 1$, a contradiction. This conclusion agrees with the answer in part (b): both say that (x_n) does not converge.

3. Let (x_n) be a sequence of real numbers. In each of the following situations, decide whether the sequence converges: if yes, give a proof why; otherwise, give an example where it does not.

(a) (10 pts) $|x_n - x_k| \le \frac{1}{n} + \frac{1}{k}$ for all $n, k \ge 1$.

Solution: Let $\epsilon > 0$, and take $N > 2/\epsilon$. Then, for all $n \ge m \ge N$, we have

$$|x_n - x_m| \le \frac{1}{n} + \frac{1}{m} \le \frac{1}{N} + \frac{1}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that (x_n) is a Cauchy sequence, and thus converges.

(b) (10 pts) For all $\epsilon > 0$, there is an $n > 1/\epsilon$ such that $|x_n| < \epsilon$.

Solution: The sequence does not necessarily converge. For instance, take $x_n = 1/n$ if n is even and $x_n = 1$ if n is odd. Then the hypothesis of the problem is satisfied: for all $\epsilon > 0$, take $n > 1/\epsilon$ and n even; then $|x_n| < \epsilon$. This shows that $\lim_{k\to\infty} x_{2k} = 0$. On the other hand, $\lim_{k\to\infty} x_{2k-1} = 1$. Therefore, the sequence (x_n) has two subsequences converging to distinct limits, and thus (x_n) does not converge.

4. Let (a_n) be a sequence and let c, d be real numbers with $c < d$. Assume that the terms in the sequence (a_n) are eventually in the closed interval $[c, d]$. Prove that

(a) (10 pts) (a_n) is bounded

Solution: Since the terms in the sequence (a_n) are eventually in the closed interval $[c, d]$, it follows that there an $N \in \mathbb{N}$ with

$$c \le a_n \le d \quad \text{for all } n \ge N \tag{1}$$

Recall that a nonempty finite set contains both a maximum and a minimum element so we have that

$$\max\{a_1, a_2, \ldots, a_{N-1}, d\}$$

is an uppper bound for (a_n), and

$$\min\{a_1, a_2, \ldots, a_{N-1}, c\}$$

is a lower bound for (a_n) and the argument is complete.

(b) (10 pts) $\liminf a_n$ and $\limsup a_n$ are elements in $[c, d]$.

Solution: The first step is to show that $\limsup a_n \le d$. From equation (1), it follows that for $n \ge N$, we have that d is an upper bound for $\{a_k : k \ge n\}$ and hence

$$v_n = \sup\{a_k : k \ge n\} \le d \quad \text{for all } n \ge N \tag{2}$$

Recall that since (a_n) is bounded the decreasing sequence v_n converges to a finite limit and the limit is $\limsup a_n$. Thus from equation (2), we have that $\limsup a_n \le d$. A similar argument shows that $c \le \liminf a_n$ and the argument is complete.

5. (10 pts) Let (a_n) be a bounded sequence, and let (a_{n_k}) be a convergent subsequence (a_n). Prove that

$$\liminf a_n \le \lim a_{n_k} \le \limsup a_n$$

Solution: The first step is to see that since the natural numbers n_k that give a subsequence of a_n have the property that

$$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$$

it follows that

$$n_k \ge k \quad \text{for all } k \in \mathbb{N}. \tag{3}$$

Equation (3) can be proved by induction as follows.

Since $n_1 \in \mathbb{N}$, we have that $n_1 \ge 1$.

For the inductive step, assume $n_k \ge k$. Then since $n_{k+1} > n_k$ and $n_k, n_{k+1} \in \mathbb{N}$ we have that $n_{k+1} \ge n_k + 1$. This together with the assumption that $n_k \ge k$ then shows that $n_{k+1} \ge k + 1$, and the proof of equation (3) is complete.

From equation (3) it follows that

$$\{a_{n_k} : k \ge n\} \subset \{a_k : k \ge n\}$$

and hence

$$\sup\{a_{n_k} : k \ge n\} \le \sup\{a_k : k \ge n\}$$

Since (a_n) is bounded, we have that (a_{n_k}) is bounded so the sequences of sups in the equation above both converge and we have

$$\lim_{k\to\infty} \sup\{a_{n_k} : k \ge n\} \le \lim_{k\to\infty} \sup\{a_k : k \ge n\}$$

The limits in the inequality above are the respective lim sups so we have

$$\limsup a_{n_k} \le \limsup a_n$$

Since (a_{n_k}) converges $\limsup(a_{n_k}) = \lim a_{n_k}$ so we have $\lim a_{n_k} \le \limsup a_n$ and the argument is complete.

6. For each of the following series, determine whether the series converges or diverges. Justify your answers.

(a) (5 pts) $\sum \dfrac{\sin(n)}{n^2}$

Solution: Since

$$\left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}$$

and $\sum(1/n^2)$ converges by the integral test, it follows from the comparison test that $\sum \sin(n)/n^2$ converges.

(b) (5 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Solution: The derivative of $\ln(\ln(x)$ equals $1/(x\ln(x))$ and $\lim_{x\to\infty} x\ln(x)$ is infinite so

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx \text{ diverges}$$

and it then follows that $\sum 1/(n\ln(n))$ diverges by the integral test.

(c) (5 pts) $\sum \frac{5n^2+6n-2}{3^n+1}$

Solution: Set $a_n = \dfrac{5n^2+6n-2}{3^n+1}$, then

$$\begin{aligned}
\lim \left|\frac{a_{n+1}}{a_n}\right| &= \lim \left|\frac{5(n+1)^2+6(n+1)-2}{5n^2+6n-2} \cdot \frac{3^n+1}{3^{n+1}+1}\right| \\
&= \lim \left|\frac{5(1+1/n)^2+6(1/n+1/n^2)-2/n^2}{5+6/n-2/n^2} \cdot \frac{1+1/3^n}{3+1/3^n}\right| \\
&= \left|\frac{5(1+0)^2+6(0+0)-0}{5+0-0} \cdot \frac{1+0}{3+0}\right| \\
&= \frac{1}{3}
\end{aligned}$$

where the second line follows from the first line by dividing both the numerator and denominator in the first line by $n^2 \cdot 3^n$. Since $\lim |a_{n+1}/a_n| = 1/3 < 1$ it follows that $\sum \dfrac{5n^2+6n-2}{3^n+1}$ converges by the ratio test.