In your work on the following problems you may use the theorems about limits in section 9 of the text.

1. (10 pts) Find a function $f(\epsilon)$ defined for $\epsilon > 0$ with the property that

$$\left|\frac{5n+6}{2n-3} - \frac{5}{2}\right| < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > f(\epsilon)$$

Solution: It suffices to show for

$$f(\epsilon) = \left(\frac{27}{4\epsilon} + \frac{3}{2}\right)$$

that given any $\epsilon > 0$

$$\left|\frac{5n+6}{2n-3} - \frac{5}{2}\right| < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > f(\epsilon)$$

This can be done as follows. First note that

$$\frac{5n+6}{2n-3} - \frac{5}{2} = \frac{10n + 12 - 10n + 15}{2(2n-3} = \frac{27}{2(2n-3)}$$

so it suffices to show that

$$\left|\frac{27}{2(2n-3)}\right| < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > f(\epsilon)$$

Now assume $n > f(\epsilon)$, then

$$\begin{aligned} n &> \left(\frac{27}{4\epsilon} + \frac{3}{2}\right) \\ 2n &> \frac{27}{2\epsilon} + 3 \\ 2n - 3 &> \frac{27}{2\epsilon} > 0 \\ \epsilon &> \frac{27}{2(2n-3)} = \left|\frac{27}{2(2n-3)}\right| \end{aligned}$$

and the proof is complete.

2. (10 pts) Find $\lim \sqrt{9n^2+2n-1} - 3n$

Solution:

$$\begin{aligned}
\lim \sqrt{9n^2+2n-1} - 3n &= \lim \sqrt{9n^2+2n-1} - 3n\left(\frac{\sqrt{9n^2+2n-1}+3n}{\sqrt{9n^2+2n-1}+3n}\right) \\
&= \lim \frac{9n^2+2n-1-9n^2}{\sqrt{9n^2+2n-1}+3n} \\
&= \lim \frac{2n-1}{\sqrt{9n^2+2n-1}+3n} \\
&= \lim \frac{2-(1/n)}{\sqrt{9+2/n-1/n^2}+3} \\
&= \frac{2-0}{\sqrt{9+0-0}+3} \\
&= \frac{2}{6} = \boxed{\frac{1}{3}}
\end{aligned}$$

where the fourth line above follows from the third line by dividing each of the the numerator and denominator in the third line by n and then moving $1/n$ into the square root expression as $1/n^2$. The fifth line follows from the fourth line using example 5 in §8 of the text, along with Theorems 9.3 and 9.6, and Lemma 9.5 and the argument is complete.

3. (10 pts) Use the $N-\epsilon$ definition of limit to show that the sequence $a_n = \sin\left(\frac{n\pi}{4}\right)$ does not converge.

Solution: Note that for $n = 8k+2$, we have $\sin\left(\frac{n\pi}{4}\right) = \sin\left(\frac{(8k+2)\pi}{4}\right) = \sin(2\pi+\pi/2) = \sin(\pi/2) = 1$, and it follows that for each $N \in \mathbb{N}$ there is an $n > N$ with $a_n = \sin\left(\frac{n\pi}{4}\right) = 1$. Similarly, by choosing values of n of the form $n = 8k-2$, it follows that for each $N \in \mathbb{N}$ there is an $n > N$ with $a_n = \sin\left(\frac{n\pi}{4}\right) = -1$. The proof now proceeds by contradiction. Suppose the sequence a_n converges to a. Then given $\epsilon = 1/2$, there is an $N \in \mathbb{N}$ with $|a_n - a| < 1/2$ for all $n > N$. From the reasoning above, we have that there is an $n_1 > N$ with $a_{n_1} = 1$ and an $n_2 > N$ with $a_{n_2} = -1$. Then using the triangle inequality we have

$$2 = |1-(-1)| = |a_{n_1} - a_{n_2}| = |a_{n_1} - a + a - a_{n_2}| \le |a_{n_1} - a| + |a - a_{n_2}| < 1/2 + 1/2 = 1$$

Since $2 < 1$ is false, the assumption that the sequence a_n converges must be false, and the proof is complete.

4. (10 pts) Prove the *Squeeze Theorem* that if $a_n \le x_n \le b_n$ for all $n \in \mathbb{N}$ and $\lim a_n = \lim b_n = L$, then the sequence x_n converges to L.

Solution: First note that directly from the definition of the absolute value (i.e., $|u| = u$ if $u \ge 0$ and $|u| = -u$ if $u < 0$) it follows that the inequality

$$|a-b| < c \quad \text{is equivalent to} \quad b-c < a < b+c \tag{1}$$

Now assume $\lim a_n = \lim b_n = L$ and an $\epsilon > 0$ has been given, then there is an $N_1 \in \mathbb{N}$ and an $N_2 \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \quad \text{for all } n > N_1$$
$$|b_n - L| < \epsilon \quad \text{for all } n > N_2$$

Thus, for $n > \max\{N_1, N_2\}$ we have $|a_n - L| < \epsilon$ and $|b_n - L| < \epsilon$ and from equation (1) and the assumption $a_n \leq x_n \leq b_n$ we have that

$$L - \epsilon < a_n \leq x_n \leq b_n < L + \epsilon \quad \text{for } n > N = \max\{N_1, N_2\}$$

From $L - \epsilon < x_n < L + \epsilon$ for $n > N$ and equation (1), we have

$$|x_n - L| < \epsilon \quad \text{for all } n > N$$

and the proof that $\lim x_n = L$ is complete.

5. Let x_n be given by $x_1 = 17$ and $x_{n+1} = \sqrt{2x_n + 15}$
   (a) (10 pts) Show that the sequence x_n is decreasing and bounded below.

   Solution: Since $x_1 = 17$, we have that $x_2 = \sqrt{2(17) + 15} = \sqrt{49} = 7$. Hence $x_2 < x_1$ and the base case for showing $x_{n+1} < x_n$ by induction is complete. For the inductive step, assume $x_{n+1} < x_n$, then

$$\begin{aligned} x_{n+1} &< x_n \\ 2x_{n+1} + 15 &< 2x_n + 15 \\ \sqrt{2x_{n+1} + 15} &< \sqrt{2x_n + 15} \\ x_{n+2} &< x_{n+1} \end{aligned}$$

   This completes the argument that the sequence x_n is decreasing.

   Note that $x_1 = 17 > 0$ and by a straightforward argument it follows by induction that if $x_n > 0$ then $x_{n+1} > 0$. Hence the sequence x_n is bounded below by 0 and the argument that the sequence x_n is decreasing and bounded below is complete.

   (b) (10 pts) Explain whether the sequence x_n converges or not. If the sequence converges, then find the limit.

   Solution: By part (a), the sequence x_n is decreasing and bounded below. Hence the sequence x_n converges by Theorem 10.2 in the text. Let $\lim x_n = x$, then using Example 5 in §8, along with Theorems 9.2 and 9.3 in the text, we have

$$\begin{aligned} \lim x_{n+1} &= \lim\left(\sqrt{2x_n + 15}\right) \\ x &= \sqrt{2 \lim x_n + 15} \\ x &= \sqrt{2x + 15} \quad \text{so} \\ x^2 &= 2x + 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x - 5)(x + 3) \end{aligned}$$

   So $x = 5$ or $x = -3$. Since the sequence is bounded below by zero and a decreasing sequence converges to its inf, we have that the limit of x_n must be greater than or equal to zero and so $x \neq -3$. Thus, $\lim x_n = 5$.

6. Consider the following definitions:
   - A sequence $\{a_n\}_{n\geq 1}$ is *eventually* in a set $A \subset \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
   - A sequence $\{a_n\}_{n\geq 1}$ is *frequently* in a set $A \subset \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

   (a) (10 pts) Is the sequence with terms $a_n = (-1)^n$ eventually or frequently in the set $\{1\}$?

   Solution: The sequence is frequently in the set $\{1\}$. Indeed, if $n = 2k$ is an even natural number, then $a_{2k} = 1$. Thus, given any $N \in \mathbb{N}$, we may take $n = N$ if N is even and $n = N + 1$ if N is odd, and then it follows that $n \geq N$ and $a_n = 1$, that is, $a_n \in \{1\}$.

   On the other hand, the sequence is *not* eventually in the set $\{1\}$. Indeed, given any $N \in \mathbb{N}$, we may take $n = N$ if N is odd and $n = N + 1$ if N is even, and then it follows that $n \geq N$ and $a_n = -1$, and so, $a_n \notin \{1\}$.

   (b) (10 pts) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

   Solution: The first definition (the sequence is *eventually* in a set) is stronger than the second definition (the sequence is *frequently* in a set).

   Indeed, suppose $\{a_n\}_{n\geq 1}$ is eventually in $A \subset \mathbb{R}$, that is, there is an $N_1 \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$. Let $N_2 \in \mathbb{N}$ be an arbitrary natural number, and put $N = \max\{N_1, N_2\}$. Then $a_n \in A$ for all $n \geq N$ (since $N \geq N_1$). In particular, $a_N \in A$ for such an integer $N \geq N_2$, thereby showing that $\{a_n\}_{n\geq 1}$ is frequently in $A \subset \mathbb{R}$.

   On the other hand, as shown in part (a), there exist sequences which are frequently but not eventually in a subset $A \subset \mathbb{R}$. Thus, the two notions are not equivalent, but rather, one (*eventually*) is stronger (or, more restrictive) than the other (*frequently*).

   (c) (10 pts) Suppose an infinite number of terms of a sequence $\{x_n\}_{n\geq 1}$ are equal to 2. Is the sequence necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

   Solution: The sequence is frequently but not necessarily eventually in the interval $(1.9, 2.1)$.

   Indeed, let $N \in \mathbb{N}$. By assumption, the set $S = \{n \in \mathbb{N} : x_n \in (1.9, 2.1)\}$ is infinite; thus, the subset $S' = \{n \in \mathbb{N} : n \geq N \text{ and } x_n \in (1.9, 2.1)\}$ is also infinite. Consequently, there is an $n \geq N$ such that $x_n \in (1.9, 2.1)$. Thus, we have shown that the sequence $\{x_n\}_{n\geq 1}$ is frequently in the interval $(1.9, 2.1)$.

   On the other hand, consider the sequence with terms $x_n = 1 + (-1)^n$. Then $x_n = 2$ for all n even and $x_n = 0$ for all n odd. Thus, arguing as in part (a), this sequence is frequently but not eventually in the interval $(1.9, 2.1)$.

   (d) (10 pts) Suppose $\lim x_n = 2$. Is the sequence $\{x_n\}_{n\geq 1}$ necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

   Solution: The sequence is eventually in the interval $(1.9, 2.1)$, and thus, by part (b), also frequently in the interval $(1.9, 2.1)$.

   To prove the first claim, recall that $\lim x_n = 2$ means the following: For every $\epsilon > 0$, there exists an $N = N(\epsilon) \in \mathbb{N}$ such that $|x_n - 2| < \epsilon$ for all $n \geq N$. Now take $\epsilon = 0.1$. There exists then an $N = N(.1) \in \mathbb{N}$ such that $|x_n - 2| < 0.1$ for all $n \geq N$. In other words, there is an $N \in \mathbb{N}$ such that $1.9 < x_n < 2.1$ for all $n \geq N$. This shows that the sequence is eventually in the interval $(1.9, 2.1)$, and the proof is complete.