1. Given a number x_n with $x_n \geq -40$ set $x_{n+1} = \sqrt{x_n + 40}$.
   (a) (10 pts) Show by mathematical induction that if $x_1 = -39$, then $x_{n+1} > x_n$ for all integers $n \geq 1$.

   Solution: Assume $x_1 = -39$ and let $P(n)$ be the statement that $x_{n+1} > x_n$. The proof that $P(n)$ is true for all $n \geq 1$ is by induction. The base case is $n = 1$.
   Given that $x_1 = -39$, we have that $x_2 = \sqrt{x_1 + 40} = \sqrt{-39 + 40} = \sqrt{1} = 1$ so $x_1 < x_2$, and hence, $P(1)$ is true.
   For the inductve step assume $P(n)$ is true. Then we have

$$\begin{aligned} x_{n+1} &> x_n \\ x_{n+1} + 40 &> x_n + 40 \\ \sqrt{x_{n+1} + 40} &> \sqrt{x_n + 40} \\ x_{(n+1)+1} &> x_{(n+1)} \end{aligned}$$

   From the last line, it follows that $P(n + 1)$ is true and the proof that $x_{n+1} > x_n$ for all $n \geq 1$ is complete.

   (b) (10 pts) Show by mathematical induction that if $x_1 = 24$, then $x_{n+1} < x_n$ for all integers $n \geq 1$.

   Solution: Assume $x_1 = 24$ and let $Q(n)$ be the statement that $x_{n+1} < x_n$. The proof that $Q(n)$ is true for all $n \geq 1$ is by induction. The base case is $n = 1$.
   Given that $x_1 = 24$, we have that $x_2 = \sqrt{x_1 + 40} = \sqrt{24 + 40} = \sqrt{64} = 8$ so $x_2 < x_1$, and hence, $Q(1)$ is true.
   For the inductve step assume $Q(n)$ is true. Then we have

$$\begin{aligned} x_{n+1} &< x_n \\ x_{n+1} + 40 &< x_n + 40 \\ \sqrt{x_{n+1} + 40} &< \sqrt{x_n + 40} \\ x_{(n+1)+1} &< x_{(n+1)} \end{aligned}$$

   From the last line, it follows that $Q(n + 1)$ is true and the proof that $x_{n+1} < x_n$ for all $n \geq 1$ is complete.

2. (15 pts) Show by induction that

$$(I(n)) \qquad \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

Solution: For $n = 1$, we have that $I(1)$ is the inequality

$$(I(1)) \qquad \frac{1}{2} \leq \frac{1}{\sqrt{3(1)+1}} = \frac{1}{2}.$$

In this case, both the inequality and the equality hold.

To procceed by induction, assume that the inequality $I(n)$ is true for $n \in \mathbb{N}$. We want to show that the inequality

$$(I(n+1)) \qquad \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\dots\cdot\frac{2n+1}{2n+2} \le \frac{1}{\sqrt{3n+4}}.$$

is true. Using the inequality $I(n)$, this is to prove

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\dots\cdot\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2} \le \frac{1}{\sqrt{3n+1}}\cdot\frac{2n+1}{2n+2} \le \frac{1}{\sqrt{3n+4}}.$$

So we want to show

$$\frac{2n+1}{2n+2} \le \frac{\sqrt{3n+1}}{\sqrt{3n+4}}.$$

By taking a square of both sides and then simplifying it is equivalent to show

$$(1) \qquad (3n+4)(4n^2+4n+1) \le (3n+1)(4n^2+8n+4).$$

The left hand side of the inequality (1) simplifies to

$$\begin{aligned}(3n+4)(4n^2+4n+1) &= 12n^3+12n^2+3n+16n^2+16n+4\\ &= 12n^3+28n^2+19n+4\end{aligned}$$

and the right hand side of the inequality (1) simplifies to

$$\begin{aligned}(3n+1)(4n^2+8n+4) &= 12n^3+24n^2+12n+4n^2+8n+4\\ &= 12n^3+28n^2+20n+4\end{aligned}$$

Thus, the inequality in equation (1) is equivalent to the inequality

$$(2) \qquad 19n \le 20n.$$

Since the inequality in equation (2) is true for all $n \ge 1$, the proof by induction that the inequality in equation $I(n)$ holds for all $n \ge 1$ is complete.

Also, since $19n < 20n$ for all $n \ge 1$, equality in $I(n)$ is not obtained for $n > 1$. Therefore, the equality is obtained if and only if $n = 1$.

3. (10 pts) Determine whether $(2+\sqrt{3})^{2/3}$ is a rational number, and explain your reasoning.

Solution: *First approach.* The first step is to see that $\sqrt{3}$ is not rational as follows. $a = \sqrt{3}$ is a solution to the equation $a^2 - 3 = 0$. By the Rational Zeros Theorem the only possible rational solutions are $\pm 1, \pm 3$. Since none of these values is a solution to $a^2 - 3 = 0$, it follows that $a^2 - 3 = 0$ does not have any rational solutions, and hence, $\sqrt{3}$ is not rational.

We can now show that $(2+\sqrt{3})^{2/3}$ is not rational as follows. Suppose $a = (2+\sqrt{3})^{2/3}$ is rational, then

$$a^3 = (2+\sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$$

is rational, and hence $(a^3 - 7)/4 = \sqrt{3}$ is rational. Thus the assumption that $a = (2+\sqrt{3})^{2/3}$ is rational implies that $\sqrt{3}$ is rational. This is a contradiction, so the assumption must be false, and it follows that $a = (2+\sqrt{3})^{2/3}$ is not rational.

Second approach. If $a = (2+\sqrt{3})^{2/3}$ is rational, then

$$\begin{aligned} a^3 &= (2+\sqrt{3})^2 = 4 + 4\sqrt{3} + 3 \quad \text{so} \\ a^3 - 7 &= 4\sqrt{3} \quad \text{hence} \\ (a^3-7)^2 &= 16(3) \quad \text{and we have} \\ a^6 - 14a^3 + 49 &= 48 \\ a^6 - 14a^3 + 1 &= 0 \end{aligned}$$

Thus, $a = (2+\sqrt{3})^{2/3}$ is a solution to $p(a) = a^6 - 14a^3 + 1 = 0$. By the Rational Zeros Theorem the only possible rational solutions are ± 1. Since neither of these is a solution to $p(a) = 0$, it follows that $p(a) = 0$ has no rational solutions, and hence $(2+\sqrt{3})^{2/3}$ is not a rational number.

4. (10 pts) Use the Rational Zeros Theorem to find all rational solutions, if any, to the equation

$$p(x) = 3x^4 - 4x^3 - x^2 - 4x - 4 = 0. \tag{3}$$

Explain your reasoning.

Solution: By the Rational Zeros Theorem, if $r = c/d$ is a solution to equation (3) where c and d are integers with no common factors, then c divides 4 and d divides 3. Thus $c = \pm 1, \pm 2, \pm 4$ and $d = \pm 1, \pm 3$, so the only possible rational solutions to equation (3) are

$$\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3$$

Using a calculator or computer gives the following values for $p(x) = 3x^4 - 4x^3 - x^2 - 4x - 4$ rounded to 1 decimal place

$p(1) = -10$	$p(2) = 0$	$p(4) = 476$
$p(-1) = 6$	$p(-2) = 80$	$p(-4) = 1020$
$p(1/3) = -5.6$	$p(2/3) = -7.7$	$p(4/3) = -11.1$
$p(-1/3) = -2.6$	$p(-2/3) = 0$	$p(-4/3) = 18.5$

Thus, there are two, and only two, rational solutions to $p(x) = 0$. The solutions are $x = 2$ and $x = -2/3$.

5. (10 pts) Show by induction using the triangle inequality that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \tag{4}$$

for all $a_i \in \mathbb{R}$ and all $n \geq 2$.

Solution: Recall the triangle inequality states that

$$|a+b| \leq |a| + |b| \quad \text{for all } a, b \in R$$

Thus the inequality (4) holds for $n = 2$.

Now assume the inequality (4) holds for a given $n \geq 2$ then

$$\begin{aligned} |a_1 + \cdots a_n + a_{n+1}| &\leq |a_1 + \cdots a_n| + |a_{n+1}| \quad \text{by (4)} \\ &\leq |a_1| + \cdots |a_n| + |a_{n+1}| \quad \text{by the inductive assumption} \end{aligned}$$

and the proof is complete.

6. Show that for $a, b \in \mathbb{R}$ we have

(a) (10 pts) $|a+b| + |a-b| = 2\max\{|a|, |b|\}$.

Solution: We have $|a+b| = \max\{a+b, -a-b\}$, $|a-b| = \max\{a-b, b-a\}$, so $|a+b|+|-a-b|$ is equal to

$$\begin{aligned}\max\{(a+b) + |a-b|, -a-b+|a-b|\} \\ &= \max\{(a+b)+(a-b), (a+b)-(a-b), \\ &\qquad -a-b+(a+b), -a-b, -a-b\} \\ &= \max\{2a, 2b, -2b, -2a\} \\ &= 2\max\{|a|, |b|\}.\end{aligned}$$

(b) (10 pts) $\big||a+b| - |a-b|\big| \le 2\min\{|a|, |b|\}$

Solution: Since $(a+b) \le (a-b) + 2b \le |a-b| + 2|b|$, we get $a+b \le |a-b| + 2|b|$. Replacing a by $-a$ and b by $-b$ gives $-a-b \le |b-a| + 2|b| = |a-b| + 2|b|$, therefore $|a+b| = \max\{a+b, -a-b\} \le |a-b| + 2|b|$ and $|a+b| - |a-b| \le 2|b|$.

Switching a and b gives $|a+b|-|a-b| = |b+a|-|b-a| \le 2|a|$, so we get $|a+b|-|a-b| \le 2\min\{|a|, |b|\}$. Replacing b by $-b$ yields

$$-(|a+b| - |a-b|) = |a-b| - |a+b| \le 2\min\{|a|, |-b|\} = 2\min\{|a|, |b|\},$$

so we conclude that

$$\big||a+b| - |a-b|\big| \le 2\min\{|a|, |b|\}.$$

7. (15 pts) Given nonempty subsets A and B of $\mathbb{R}$, define the set $A - B$ by

$$A - B = \{a - b \colon a \in A, b \in B\}.$$

State and prove a formula for $\inf(A-B)$ in terms of $\inf(A), \sup(A), \inf(B)$, and $\sup(B)$.

Solution: We will show that

$$\inf(A-B) = \inf(A) - \sup(B). \tag{5}$$

Set $a := \inf(A)$ and $b := \sup(B)$. We prove (5) by first showing that $\inf(A-B) \ge a - b$, and then showing that $a - b \ge \inf(A-B)$.

To prove the first inequality, let $x \in A$ and $y \in B$. By the definition of $a = \inf(A)$, we have that $x \ge a$. Likewise, by the definition of $b = \sup(B)$, we have that $y \le b$, or, $-y \ge -b$. Therefore,

$$x - y \ge a - b.$$

Since any element in the set $A - B$ is of the form $x - y$, for some $x \in A$ and $y \in B$, it follows from the definition of infimum that $\inf(A-B) \ge a - b$.

To prove the second inequality, let $\varepsilon > 0$. By the definition of $a = \inf(A)$, there in an element $x \in A$ such that $a + \varepsilon/2 \ge x$. Likewise, by the definition of $b = \sup(B)$, there in an element $y \in B$ such that $b - \varepsilon/2 \le y$, or, $-b + \varepsilon/2 \ge -y$. Therefore, $(a + \varepsilon/2) + (-b + \varepsilon/2) \ge x - y$, or,

$$a - b + \varepsilon \ge x - y.$$

Using now the definition of $\inf(A-B)$, we infer that

$$a - b + \varepsilon \ge \inf(A-B).$$

But this inequality holds for any $\varepsilon > 0$, we conclude that

$$a - b \ge \inf(A-B),$$

and this completes the proof.