Problem 1. Let \((s_n)\) be a sequence that converges
(a) Show that if \(s_n \geq a\) for all but finitely many \(n\), then \(\lim s_n \geq a\).
(b) Show that if \(s_n \leq b\) for all but finitely many \(n\), then \(\lim s_n \leq b\).
(c) Conclude that if all but finitely many \(s_n\) belong to \([a, b]\), then \(\lim s_n \in [a, b]\).

Solution.
(a) Let \(m\) be the largest integer such that \(s_m < a\) and let \(s = \lim s_n\). Proceeding by contradiction, suppose that \(s < a\). Choose \(\epsilon\) such that \(0 < \epsilon < a - s\). Since \(s_n \to s\), there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\),
\[ s_n < s + \epsilon < s + a - s = a. \]
In particular, this holds for \(n > \max \{N, m\}\), but then \(s_n < a\) contradicts maximality of \(m\).
(b) Let \(m\) be the smallest integer such that \(s_m > b\) and let \(s = \lim s_n\). Proceeding by contradiction, suppose that \(s > b\). Choose \(\epsilon\) such that \(0 < \epsilon < s - b\). Since \(s_n \to s\), there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\),
\[ b = s - (s - b) < s - \epsilon < s_n. \]
In particular, this holds for \(n > \max \{N, m\}\), but then \(s_n > b\) contradicts maximality of \(m\).
(c) By part (a), \(s = \lim s_n \geq a\) and by part (b) \(s \leq b\), so \(s \in [a, b]\).

Problem 2. Let \(x_1 = 1\) and \(x_{n+1} = 3x_n^2\) for \(n \geq 1\).
(a) Show if \(a = \lim x_n\), then \(a = \frac{1}{3}\) or \(a = 0\).
(b) Does \(\lim x_n\) exist? Explain.
(c) Discuss the apparent contradiction between parts (a) and (b).

Solution.
(a) Suppose \(a = \lim x_n\) exists. Then invoking the limit theorem for the identity \(x_{n+1} = 3x_n^2\) gives
\[ \lim_{n \to \infty} x_{n+1} = 3(\lim_{n \to \infty} x_n)^2 \implies a = 3a^2. \]
The only two solutions to this equation are \(a = 0\) or \(a = \frac{1}{3}\).
(b) The limit does not exist. In fact, we can show \(x_{n+1} \geq 3^n\) for all \(n\) (or a lower bound which grows even more quickly if we want). Indeed, \(x_2 = 3\), and by induction, \(x_{n+1} = 3x_n^2 \geq 3(3^{n-1})^2 = 3^{2n-1} \geq 3^n\).
Since \(3^n\) diverges to infinity, it follows that \(x_n\) must also.
(c) The application of the limit theorem \(\lim(x_n^2) = (\lim x_n)^2\) in part (a) is only valid in case that \(\lim x_n\)
is a finite real number.

Problem 3. Assume all \(s_n \neq 0\) and the limit \(L = \lim \left| \frac{s_{n+1}}{s_n} \right|\) exists.
(a) Show that if \(L < 1\), then \(\lim s_n = 0\).
(b) Show that if \(L > 1\), then \(\lim |s_n| = +\infty\).

Solution.
(a) Define the sequence \(r_n = \left| \frac{s_{n+1}}{s_n} \right|\) of positive real numbers, and suppose that \(\lim r_n = L < 1\). Choose \(a \in \mathbb{R}\) such that \(L < a < 1\), and let \(\epsilon = a - L\). Since \(r_n \to L\), there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\),
\[ r_n < L + \epsilon = a. \]
This implies \( |s_{n+1}| < a |s_n| \) for all \( n \geq N \), and in particular \( |s_{N+1}| < a |s_N| \). This is the base case for an induction, where \( |s_{N+k}| < a^k |s_N| \) implies \( |s_{N+k+1}| < a |s_{N+k}| < a^{k+1} |s_N| \), which may be rewritten as the statement \( |s_n| < a^{n-N} |s_N| \) for all \( n > N \). We therefore have \[
0 \leq |s_n| \leq c a^n \quad \forall \ n > N,
\]
where \( c = \frac{|s_N|}{a^N} \) is a constant. Since \( a < 1 \), the sequence \( a^n \) converges to 0, and \( c \cdot a^n \to 0 \) also. By the squeeze lemma, it follows that \( |s_n| \to 0 \) which implies \( s_n \to 0 \).

(b) Define the sequence \( t_n = \frac{1}{|s_n|} \). Then supposing that \( \lim \frac{s_{n+1}}{s_n} = L > 1 \), it follows that \( \lim \frac{t_{n+1}}{t_n} = L^{-1} < 1 \). By part (a), \( \lim t_n = 0 \), and by Theorem 9.10, it follows that \( \lim |s_n| = +\infty \). \( \square \)

**Problem 4.**

(a) Let \((s_n)\) be a sequence in \( \mathbb{R} \) such that

\[
|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}.
\]

Prove that \((s_n)\) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume \( |s_{n+1} - s_n| < \frac{1}{n} \) for all \( n \in \mathbb{N} \)?

**Solution.**

(a) Let \( n, k \in \mathbb{N} \). Consider \( |s_{n+k} - s_n| \). Adding and subtracting \( s_{n+k-1}, s_{n+k-2}, \ldots, s_{n+1} \) and employing the triangle inequality, we have

\[
|s_{n+k} - s_n| \leq |s_{n+k} - s_{n+k-1}| + |s_{n+k-1} - s_{n+k-2}| + \cdots + |s_{n+1} - s_n| < 2^{-n} + 2^{-(n+1)} + \cdots + 2^{-(n+k-1)}.
\]

Using the identity \( 1 + r + \cdots + r^l = \frac{1-r^{l+1}}{1-r} \) for \( r < 1 \) in the case \( r = \frac{1}{2}, \ l = k - 1 \), we have

\[
2^{-n} + \cdots + 2^{-(n+k-1)} = 2^{-n} \frac{1+2^{-k}}{1/2} < 2^{-n} \frac{1}{1/2} = 2^{-n+1},
\]

thus

\[
(1) \quad |s_{n+k} - s_n| < 2^{-n+1}.
\]

To prove that \( s_n \) is Cauchy, given \( \varepsilon > 0 \) choose \( N \in \mathbb{N} \) such that \( 2^{-N+1} < \varepsilon \). (This is possible since \( 2^{-N+1} \to 0 \) as \( n \to \infty \).) Then for any pair \( m, n \geq N \), (without loss of generality, \( m \geq n \) so \( m = n + k \) for some \( k \geq 0 \),

\[
|s_m - s_n| = |s_{n+k} - s_n| < 2^{-n+1} \leq 2^{-N+1} < \varepsilon.
\]

Since \((s_n)\) is Cauchy and \( \mathbb{R} \) is complete, we conclude that \((s_n)\) converges.

(b) The result is false if we only assume \( |s_{n+1} - s_n| < \frac{1}{n} \). As a counter-example, let \( s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) (the partial summations of the harmonic series). Then \( |s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n} \), but the sequence \((s_n)\) diverges to infinity. (One way to see this is as follows:

\[
s_{2k} = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \cdots + \frac{1}{8} \right) + \left( \frac{1}{9} + \cdots + \frac{1}{16} \right) + \cdots + \left( \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k} \right)
\]

\[
\geq 1 + \left( \frac{1}{2} + \frac{1}{4} \right) + \left( \frac{1}{8} + \cdots + \frac{1}{16} \right) + \left( \frac{1}{16} + \cdots + \frac{1}{32} \right) + \cdots + \left( \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k} \right)
\]

\[
= 1 + \left( \frac{1}{2} + \frac{1}{4} \right) + \left( \frac{8}{16} \right) + \cdots + \frac{2^{k-1}}{2^k} = \frac{k + 2}{2}.
\]

Given any \( M > 0 \) we can choose a \( k \) such that \( \frac{k+2}{2} > M \), and so \( s_n > M \) for \( n = 2^k \); hence \( s_n \to +\infty \). \( \square \)

**Problem 5.** Let \( s_1 = 1 \) and \( s_{n+1} = \frac{1}{3} (s_n + 1) \) for \( n \geq 1 \).

(a) Find \( s_2, s_3 \) and \( s_4 \).
(b) Use induction to show \( s_n > \frac{1}{2} \) for all \( n \).
(c) Show \((s_n)\) is a decreasing sequence.
(d) Show \( \lim s_n \) exists and find \( \lim s_n \).

**Solution.**
(a) \( s_2 = \frac{2}{3}, \ s_3 = \frac{5}{9}, \ s_4 = \frac{14}{27} \).
(b) \( s_1 = 1 > \frac{1}{2} \) holds. By induction, supposing that \( s_n > \frac{1}{2} \), we have
\[
 s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2},
\]
so \( s_n > \frac{1}{2} \) for all \( n \).
(c) Let \( r_n = s_n - s_{n+1} \). We will show by induction that \( r_n \geq 0 \) for all \( n \). We have \( r_1 = 1 - \frac{2}{3} = \frac{1}{3} > 0 \).
Assuming \( r_n \geq 0 \),
\[
r_{n+1} = s_n - s_{n+1} = \frac{1}{3}((s_{n-1} + 1) - (s_n + 1)) = \frac{1}{3}(s_{n-1} - s_n) = \frac{1}{3}r_n \geq 0,
\]
completing the inductive step. Thus \((s_n)\) is decreasing.
Alternatively, (not using induction),
\[
 s_n > \frac{1}{2} \implies \frac{2}{3}s_n > \frac{1}{3} \implies \frac{1}{3}(s_n + 1) < s_n \implies s_{n+1} < s_n,
\]
which holds for all \( n \) by the previous part.
(d) Since \((s_n)\) is a decreasing sequence which is bounded below, it converges to some \( s = \lim s_n \). Using the limit theorem,
\[
 \lim s_{n+1} = \frac{1}{3}(\lim s_n + 1) \implies s = \frac{1}{3}(s + 1) \implies s = \frac{1}{2}.
\]

**Problem 6.** Let \((s_n)\) be the sequence of numbers in Fig. 11.2 in the book.
(a) Find the set \( S \) of subsequential limits of \((s_n)\).
(b) Determine \( \lim \sup s_n \) and \( \lim \inf s_n \).

**Solution.**
(a) We claim that \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\} \). Indeed, for any \( \frac{1}{n} \), there are infinitely many \( k \in \mathbb{N} \) such that \( s_k = \frac{1}{n} \), which implies that \((s_n)\) has a constant subsequence \((\frac{1}{n}, \frac{1}{n}, \ldots)\). In the case of 0, for any \( \varepsilon > 0 \), there are infinitely many \( s_k \) such that \( |s_k - 0| < \varepsilon \); indeed, we may take \( n \) such that \( \frac{1}{n} < \varepsilon \) and consider the constant subsequence \((\frac{1}{n}, \frac{1}{n}, \ldots)\) again. There are no other subsequential limits.
(b) \( \lim \inf s_n = \inf S = 0 \) and \( \lim \sup s_n = \sup S = 1 \).