Homework # 1, Real Analysis

P35. 5. $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ is such a field.

Suppose it's an ordered field.

(i) Suppose $1 \leq 0$. Then $1+1 \leq 0+1 \leq 0$, i.e. $2 \leq 0$

So $0 = 2+1 \leq 0+1 = 1$. Contradiction

(ii) Suppose $0 \leq 1$. Then $0 \leq 2$, so $1 \leq 0$. Contradiction

P45 1. a. $x_n = \sqrt{x_{n-1}+2}$ Take limits of both sides $\lambda = \sqrt{\lambda+2}$

Hence $\lambda^2 - \lambda - 2 = 0$

b. $\lambda = 2$ or -1. The sequence is non-negative and increasing

So the limit is non-negative. Hence $\lambda = 2$

2. The sequence is positive.

And for $n \geq 6$ $a_n = a_6 \cdot \frac{3}{7} \cdot \frac{3}{8} \cdot \cdots \cdot \frac{3}{n} < a_6 \cdot \left(\frac{1}{2}\right)^{n-6}$

So as $n \to \infty$ $a_n \to 0$.

3. $x_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} \to 0$ as $n \to \infty$

4. Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then $x_{n+1} - x_n = \frac{1}{n+1} < \frac{1}{n}$.

But $\{x_n\}$ is the harmonic sequence which is divergent.

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3. (a) If $x > 0$. Let $\varepsilon = \frac{x}{2}$ then $0 < \varepsilon < x$. contradiction.

Hence $x = 0$

(b) Let $x = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$

2. Pf. Suppose not, $\sqrt{k} = \frac{p}{q}$ $\gcd(p,q) = 1$. $k = \frac{p^2}{q^2}$ so $q^2 | p^2 \Rightarrow q | p$

So $q = \gcd(p,q) = 1$. Hence $k = p^2$ contrdiction.

33. (a) By L'Hopital's rule $\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$.

(b) $\ln n^{1/n} = \frac{\ln n}{n} \to 0$ as $n \to \infty$ Hence $n^{1/n} \to 1$ as $n \to \infty$.