1. [10 points] Solve the following initial value problems.

(a) \((1 + 2t) \frac{dy}{dt} = 1 + y^2, \ y(0) = 0.\)

(b) \(\frac{dy}{dt} + 2 \frac{y}{t} = t, \ y(1) = -1.\)
2. [8 points] Consider the initial-value problem \( \frac{dy}{dt} = y^3, \ y(0) = \frac{1}{4} \).

(a) Find a solution \( y = y(t) \).

(b) Give the domain of definition of the solution.

(c) What happens to the solution as it approaches the upper end of the domain of definition?
3. [12 points] An 80 gallon tank initially contains 20 gallons of pure water. A brine solution containing 6 pounds of salt per gallon enters the tank at 4 gallons per minute, and the mixture is removed at the rate of 2 gallons per minute.

(a) Write down the initial value problem that describes the quantity of salt (in pounds) at time \( t \) (in minutes).

(b) Solve this initial value problem.

(c) When will the tank start to overflow? How many pounds of salt will there be in the tank at that time?
4. [10 points] Solve the following initial value problems.

(a) $y'' - y = e^t, \quad y(0) = 0, \quad y'(0) = 0$

(b) $y'' + y = e^t, \quad y(0) = 0, \quad y'(0) = 0$
5. [8 points] Consider the linear system $\mathbf{Y}' = A\mathbf{Y}$, with $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

(a) It turns out that $A$ has a single eigenvalue, $\lambda$. Compute this eigenvalue.

(b) Given the initial condition $\mathbf{V}_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find the corresponding eigenvector $\mathbf{V}_1$ for the eigenvalue $\lambda$.

(c) Find the solution $\mathbf{Y} = \mathbf{Y}(t)$ with initial condition $\mathbf{Y}(0) = \mathbf{V}_0$. 
6. Consider the linear system $\mathbf{Y}' = A\mathbf{Y}$, with $A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $A$.

(b) Find the solution $\mathbf{Y} = \mathbf{Y}(t)$ of the system, with initial condition $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Sketch the phase portrait, indicating the straight-line solutions, and at least 4 other solution curves. Put arrows on the curves to show their direction. What kind of equilibrium point is the origin?
7. 8 points \[ \text{Consider the one-parameter family of linear systems } \mathbf{X}' = A \mathbf{X}, \text{ where the matrix } A = \begin{pmatrix} a & a \\ -1 & 0 \end{pmatrix} \text{ depends on the parameter } a. \]

(a) Compute the trace \( T \), the determinant \( D \). Sketch the corresponding curve in the \( T-D \) plane.

(b) Find the bifurcation values for \( a \) (i.e., the values of \( a \) for which the nature of the phase plane for the linear system changes).

(c) For each typical value of \( a \) in between or outside bifurcation points, indicate in words the nature of the corresponding phase plane.
Consider the system
\[\begin{align*}
\frac{dx}{dt} &= (x - 2)(y - 1) \\
\frac{dy}{dt} &= y(y + x - 1)
\end{align*}\]

(a) Find the equilibrium points.

(b) Find the Jacobian matrix of the system.

(c) Find the linearized system for each of the equilibrium points from part (a). In each case, compute the eigenvalues of the corresponding matrix, and classify the equilibrium point as either source, sink, saddle point, center, etc.
9. [10 points]

(a) Find the Laplace transform \( F(s) = \mathcal{L}[f(t)] \) of the function

\[
f(t) = \begin{cases} 
0, & t < 2 \\
t + e^{2t-3}, & t \geq 2 
\end{cases}
\]

(b) Find the inverse Laplace transform \( f(t) = \mathcal{L}^{-1}[F(s)] \) of the function

\[
F(s) = \frac{1}{s + 1} + \frac{s - 5}{s^2 + 6s + 13}
\]
10. [12 points] Use Laplace transforms to solve the following initial value problems.

(a) $y' - 2y = 4 + \delta_1(t), \quad y(0) = 1$

(b) $y'' - 3y' + 2y = 6, \quad y(0) = 0, \quad y'(0) = 1$