A Lefschetz hyperplane theorem with an assigned base point

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“If the sum of the Milnor numbers at the singular points of $V(h)$ is large, then $V(h)$ cannot have a point of large multiplicity, unless $V(h)$ is a cone.”
Notations:

- \( h \in \mathbb{C}[z_0, \ldots, z_n] \) is a homogeneous polynomial of degree \( d \).
- \( V(h) := \{ h = 0 \} \subseteq \mathbb{P}^n \) is the projective hypersurface defined by \( h \).
- \( D(h) := \{ h \neq 0 \} \subseteq \mathbb{P}^n \) is the smooth affine variety defined by \( h \).
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- \( D(h) := \{ h \neq 0 \} \subseteq \mathbb{P}^n \) is the smooth affine variety defined by \( h \).
- The **gradient map** of \( h \) is the rational map

\[ \text{grad}(h) : \mathbb{P}^n \to \mathbb{P}^n, \quad z \mapsto \left( \frac{\partial h}{\partial z_0} : \cdots : \frac{\partial h}{\partial z_n} \right). \]

- The **polar degree** of \( h \) is the degree of \( \text{grad}(h) \).
If $V(h)$ has only isolated singular points, then

$$\deg(\text{grad}(h)) = (d - 1)^n - \sum_{p \in V(h)} \mu^{(n)}(p),$$

where $\mu^{(n)}(p)$ is the Milnor number of $V(h)$ at $p$. 
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$$\deg(\text{grad}(h)) = (d - 1)^n - \sum_{p \in V(h)} \mu^{(n)}(p),$$

where $\mu^{(n)}(p)$ is the Milnor number of $V(h)$ at $p$.

**Theorem (A)**

Suppose $V(h)$ has only isolated singular points, and let $m$ be the multiplicity of $V(h)$ at one of its points $x$. Then

$$\deg(\text{grad}(h)) \geq (m - 1)^{n-1},$$

unless $V(h)$ is a cone with the apex $x$. 
Theorem (A)

Suppose \( V(h) \) has only isolated singular points, and let \( m \) be the multiplicity of \( V(h) \) at one of its points \( x \). Then

\[
\deg(\text{grad}(h)) \geq (m - 1)^{n-1},
\]

unless \( V(h) \) is a cone with the apex \( x \).

It is interesting to observe how badly the inequality fails when \( V(h) \) is a cone over a smooth hypersurface in \( \mathbb{P}^{n-1} \subseteq \mathbb{P}^n \).

In this case, the polar degree is zero, but the apex of the cone has multiplicity \( d \).
Theorem (B)

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n - 1)$-th sectional Milnor number of $V(h)$ at one of its points $x$. Then

$$\text{deg} \left( \text{grad}(h) \right) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex $x$. 

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Theorem (B)

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n - 1)$-th sectional Milnor number of $V(h)$ at one of its points $x$. Then

$$\deg(\text{grad}(h)) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex $x$.

A theorem of Teissier says that, locally at any point $x$,

$$\frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \frac{\mu^{(n-1)}}{\mu^{(n-2)}} \geq \cdots \geq \frac{\mu^{(i)}}{\mu^{(i-1)}} \geq \cdots \geq \frac{\mu^{(1)}}{\mu^{(0)}}.$$

Therefore Theorem (B) implies Theorem (A).
Theorem (B)

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n - 1)$-th sectional Milnor number of $V(h)$ at one of its points $x$. Then

$$\deg(\text{grad}(h)) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex $x$.

The inequality of Theorem (B) is tight relative to the degree and the dimension:

For each $d \geq 3$ and $n \geq 2$, there is a degree $d$ hypersurface in $\mathbb{P}^n$ with one singular point, for which the equality holds in Theorem (B).
Conjecture (Dimca and Papadima ’03)

A projective hypersurface with only isolated singular points has polar degree 1 if and only if it is one of the following, after a linear change of coordinates:

- \((n \geq 2, d = 2)\) a smooth quadric
  \[
  h = z_0^2 + \cdots + z_n^2 = 0.
  \]

- \((n = 2, d = 3)\) the union of three nonconcurrent lines
  \[
  h = z_0 z_1 z_2 = 0.
  \]

- \((n = 2, d = 3)\) the union of a smooth conic and one of its tangent
  \[
  h = z_0 (z_1^2 + z_0 z_2) = 0.
  \]
Theorem (C)

The conjecture of Dimca and Papadima is true.
I will sketch an argument for the above theorems when \( n \geq 3 \),
using the *Lefschetz hyperplane theorem with an assigned base point*.

Interestingly, our proof does not work for plane curves.

For \( n = 2 \), one has to argue separately.

(In this case the above statements are theorems of Dolgachev).
Part 2. Lefschetz theorem with an assigned base point

“We may assign a base point when applying Lefschetz hyperplane theorem (unless our variety has a special geometry with respect to the base point).”

“This extra freedom enables us to relate local and global invariants of the variety.”
We drop the assumption that $V(h)$ has only isolated singularities.

Hamm’s Lefschetz theory shows that, if $H$ is a general hyperplane in $\mathbb{P}^n$, then

$$\pi_i\left(D(h), D(h) \cap H\right) = 0 \quad \text{for} \quad i < n.$$ 

We refine this result by allowing hyperplanes to have an assigned base point.
Theorem (D)

If $H_x$ is a general hyperplane passing through a point $x$ in $\mathbb{P}^n$, then

$$\pi_i(D(h), D(h) \cap H_x) = 0 \quad \text{for} \quad i < n,$$

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1. one of the components of $V(h)$ is a cone with the apex $x$, or
2. the singular locus of $V(h)$ contains a line passing through $x$. 
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1. one of the components of $V(h)$ is a cone with the apex $x$, or
2. the singular locus of $V(h)$ contains a line passing through $x$.

Since $D(h)$ and $D(h) \cap H_x$ are homotopic to CW-complexes of dimensions $n$ and $n - 1$ respectively, the vanishing of the homotopy groups implies

$$H_i\left( D(h), D(h) \cap H_x \right) = 0 \quad \text{for} \quad i \neq n.$$
An example showing that the first condition is necessary for the conclusion:

Example

Let $V(h)$ be the plane curve consisting of a nonsingular conic containing $x$, the tangent line to the conic at $x$, and a general line passing through $x$. Then

$$H_1(D(h), D(h) \cap H_x) \cong H_1(S^1 \times S^1, S^1) \cong \mathbb{Z} \neq 0.$$
How do we prove something like Theorem (D)?

We go back to the idea of Lefschetz.
Say $X$ is a smooth projective variety of dimension $n$, and let $A$ be a general codimension 2 linear subspace of a fixed ambient projective space of $X$. 
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The main conclusion of Lefschetz is the isomorphism

\[
H_{i+1}(X, X_c) \sim H_{i-1}(X_c, X_c \cap A), \quad i < n - 1,
\]

where \( X_c \) is a general member of the pencil of hyperplane sections of \( X \) associated to \( A \).

(By induction, one has the vanishing \( H_i(X, X_c) = 0 \) for \( i < n \).)
Why does he need $A$ to be general?

Let $\mathcal{P}_A$ be the pencil of hyperplanes associated to $A$, and let $\tilde{X}$ be the blowup of $X$ along $X \cap A$. 
Why does he need \( A \) to be general?

Let \( \mathcal{P}_A \) be the pencil of hyperplanes associated to \( A \), and let \( \tilde{X} \) be the blowup of \( X \) along \( X \cap A \).

The point of the genericity is that, for such \( A \), the map

\[
p : \tilde{X} \longrightarrow \mathcal{P}_A \sim \mathbb{P}^1
\]

has only isolated singular points.
This main idea has been refined in the last ninety years.

Here is the current version (which we need).
• $Y$ is a projective variety.
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- $X$ is the quasi-projective variety $Y \setminus V$. 

$W$ is a Whitney stratification of $Y$ such that $V$ is a union of strata.

$A$ is a codimension 2 linear subspace of a fixed ambient projective space of $Y$.

$W_j Y_n A$ is the Whitney stratification of $Y_n A$ obtained by restricting $W$.

$P_A$ is the pencil of hyperplanes containing the axis $A$. We write $Y_n A \to P_A$ for the map sending $p$ to the member of $P_A$ containing $p$.

$e Y$ is the blow-up of $Y$ along $Y \setminus A$. We write $p : e Y \to P_A$ for the map which agrees with $p$ on $Y_n A$. 

$S$ is a Whitney stratification of $e Y$ which extends $W_j Y_n A$. 

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\begin{itemize}
  \item $Y$ is a projective variety.
  \item $V$ is a closed subset of $Y$.
  \item $X$ is the quasi-projective variety $Y \setminus V$.
  \item $\mathcal{W}$ is a Whitney stratification of $Y$ such that $V$ is a union of strata.
\end{itemize}
- \( Y \) is a projective variety.
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- $W$ is a Whitney stratification of $Y$ such that $V$ is a union of strata.
- $A$ is a codimension 2 linear subspace of a fixed ambient projective space of $Y$.
- $W|_{Y \setminus A}$ is the Whitney stratification of $Y \setminus A$ obtained by restricting $W$. 
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- \( W|_{Y \setminus A} \) is the Whitney stratification of \( Y \setminus A \) obtained by restricting \( W \).
- \( \mathcal{P}_A \) is the pencil of hyperplanes containing the axis \( A \). We write
  \[ \pi : Y \setminus A \to \mathcal{P}_A \]
  for the map sending \( p \) to the member of \( \mathcal{P}_A \) containing \( p \).
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  \item $\widetilde{Y}$ is the blow-up of $Y$ along $Y \cap A$. We write
    \[ p : \widetilde{Y} \to \mathcal{P}_A \]
    for the map which agrees with $\pi$ on $Y \setminus A$.
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- $Y$ is a projective variety.
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- $A$ is a codimension 2 linear subspace of a fixed ambient projective space of $Y$.
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- $P_A$ is the pencil of hyperplanes containing the axis $A$. We write
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  for the map sending $p$ to the member of $P_A$ containing $p$.
- $\widetilde{Y}$ is the blow-up of $Y$ along $Y \cap A$. We write
  \[ p : \widetilde{Y} \to P_A \]
  for the map which agrees with $\pi$ on $Y \setminus A$.
- $S$ is a Whitney stratification of $\widetilde{Y}$ which extends $W|_{Y \setminus A}$.
The singular locus of $p$ with respect to $\mathcal{I}$ is the following closed subset of $\tilde{Y}$:

$$\text{Sing}_{\mathcal{I}} p := \bigcup_{s \in \mathcal{I}} \text{Sing} p|_{s}.$$ 

We say that $\mathcal{P}_{A}$ has only only isolated singular points with respect to $\mathcal{I}$ if

$$\dim \text{Sing}_{\mathcal{I}} p \leq 0.$$ 

The singular locus of $p$ is a closed subset of $\tilde{Y}$ because $\mathcal{I}$ is a Whitney stratification.
Theorem (Lefschetz, Andreotti, Frankel, Hamm, Lê, Deligne, Goresky, MacPherson, Nemethi, Siersma, Tibăr)

Let $X_c$ be a general member of the pencil on $X$. Suppose that

1. the axis $A$ is not contained in $V$,

2. the rectified homotopical depth of $X$ is $\geq n$ for some $n \geq 2$,

3. the pencil $\mathcal{P}_A$ has only isolated singular points with respect to $\mathcal{I}$, and

4. the pair $(X_c, X_c \cap A)$ is $(n - 2)$-connected.

Then the pair $(X, X_c)$ is $(n - 1)$-connected.

Replace the condition 2 by “$\dim X = n \geq 2$ and $X$ is a local complete intersection” if you don’t like the rectified homotopical depth.
We are ready for the \textit{inductive} proof of

\begin{center}
\textbf{Theorem (D)}
\end{center}

\textit{If $H_x$ is a general hyperplane passing through a point $x$ in $\mathbb{P}^n$, then}

\[ \pi_i(D(h), D(h) \cap H_x) = 0 \text{ for } i < n, \]

\textit{unless}

1. \textit{one of the components of $V(h)$ is a cone with the apex $x$, or}
2. \textit{the singular locus of $V(h)$ contains a line passing through $x$.}
We are ready for the *inductive* proof of

**Theorem (D)**

*If $H_x$ is a general hyperplane passing through a point $x$ in $\mathbb{P}^n$, then*

$$\pi_i(D(h), D(h) \cap H_x) = 0 \quad \text{for} \quad i < n,$$

*unless*

1. one of the components of $V(h)$ is a cone with the apex $x$, or
2. the singular locus of $V(h)$ contains a line passing through $x$.

Let $A_x$ be a general codimension 2 linear subspace of $\mathbb{P}^n$ containing $x$,

and let $\widetilde{\mathbb{P}}^n$ be the blowup of $\mathbb{P}^n$ along $A_x$. 
Our goal is

a. to show that the two conditions on $V(h)$ are satisfied by $V(h) \cap H_x$, where $H_x$ is a general member of the pencil $\mathcal{P}_{A_x}$,
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where \( H_x \) is a general member of the pencil \( \mathcal{P}_{A_x} \),

b. to produce a Whitney stratification \( \mathcal{S} \) of \( \mathbb{P}^n \) such that

i. \( V(h) \setminus A_x \) is a union of strata,

ii. the map

\[
p : \mathbb{P}^n \longrightarrow \mathcal{P}_{A_x}
\]

has only isolated singularities with respect to \( \mathcal{S} \),

and
Our goal is

a. to show that the two conditions on $V(h)$ are satisfied by $V(h) \cap H_x$, where $H_x$ is a general member of the pencil $\mathcal{P}_{A_x}$,

b. to produce a Whitney stratification $\mathcal{I}$ of $\tilde{\mathbb{P}}^n$ such that
   i. $V(h) \setminus A_x$ is a union of strata,
   ii. the map

   \[ p : \tilde{\mathbb{P}}^n \longrightarrow \mathcal{P}_{A_x} \]

   has only isolated singularities with respect to $\mathcal{I}$,

   and

c. to check for $n = 2$, which is an assertion on the fundamental group of plane curve complements.
Of course, we have to use our conditions on $V(h)$ at some point, since otherwise $a,b,c$ are not possible.
Let $V$ be an irreducible subvariety of positive dimension $k + 1$ in $\mathbb{P}^n$.

**Lemma (a)**

The following conditions are equivalent for a point $x$ in $\mathbb{P}^n$.

1. $V$ is a cone with the apex $x$.

2. For any point $y$ of $V$ different from $x$, the line joining $x$ and $y$ is contained in $V$.

3. If $E_x$ is a general codimension $k$ linear subspace in $\mathbb{P}^n$ containing $x$, then every irreducible component of $V \cap E_x$ is a line containing $x$.

The irreducibility assumption is clearly necessary in order to deduce 3 from 4.
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3. If $E_x$ is a general codimension $k$ linear subspace in $\mathbb{P}^n$ containing $x$, then every irreducible component of $V \cap E_x$ is a line containing $x$.

4. If $E_x$ is a general codimension $k$ linear subspace in $\mathbb{P}^n$ containing $x$, then some irreducible component of $V \cap E_x$ is a line containing $x$.

The irreducibility assumption is clearly necessary in order to deduce 3 from 4.
Here is another characterization of cones, in the view point of Lefschetz theory.

Let $S$ be a smooth and irreducible locally closed subset of $\mathbb{P}^n$.

($S$ will be a stratum of the stratification $\mathcal{S}$.)

**Lemma (b)**

*If $A_x$ is a general codimension 2 linear subspace passing through a point $x$ in $\mathbb{P}^n$, then

$$p_{A_x} : S \setminus A_x \rightarrow \mathcal{P}_{A_x}$$

has only isolated singular points, unless the closure of $S$ in $\mathbb{P}^n$ is a cone with the apex $x$.***
Suppose that

- no component of $V(h)$ is a cone over a smooth variety with the apex $x$, and
- the singular locus of $V(h)$ does not contain a line passing through $x$.

Then we can find a Whitney stratification $\mathcal{W}$ of $\mathbb{P}^n$ such that

- $\{x\}$ is a stratum of $\mathcal{W}$,
- $V(h)$ is a union of strata of $\mathcal{W}$, and
- the closure of a stratum of $\mathcal{W} \setminus \{\{x\}\}$ is not a cone with the apex $x$.

$\mathbb{P}^n$ is a subset of $\mathbb{P}^n \times \mathbb{P}^1$. 
Suppose that

- no component of \( V(h) \) is a cone over a smooth variety with the apex \( x \), and
- the singular locus of \( V(h) \) does not contain a line passing through \( x \).

Then we can find a Whitney stratification \( W \) of \( \mathbb{P}^n \) such that

- \( \{x\} \) is a stratum of \( W \),
- \( V(h) \) is a union of strata of \( W \), and
- the closure of a stratum of \( W \setminus \{\{x\}\} \) is not a cone with the apex \( x \).

\( \mathbf{P}^n \) is a subset of \( \mathbb{P}^n \times \mathbb{P}^1 \). We use \( W \) to produce the stratification \( \mathcal{S} \) of \( \mathbf{P}^n \).
Lemma (b')

Let $\mathcal{S}$ be the stratification of $\mathbb{P}^n$ with strata

1. $(S \times \mathbb{P}^1) \cap (\mathbb{P}^n \setminus A \times \mathbb{P}^1)$ for $S \in \mathcal{W} \setminus \{\{x\}\}$,

2. $(S \times \mathbb{P}^1) \cap (A \times \mathbb{P}^1)$ for $S \in \mathcal{W} \setminus \{\{x\}\}$,

3. $\{x\} \times \mathbb{P}^1 \setminus E$, and

4. $E$,

where $E$ is the set of points at which one of the strata from (1) and (2) fails to be Whitney regular over $\{x\} \times \mathbb{P}^1$. Then, for a sufficiently general $A$ through $x$,

1. $\mathcal{S}$ is a Whitney stratification, and

2. $\mathcal{P}$ has only isolated singular points with respect to $\mathcal{S}$. 

Now the base case of the induction.

Let $C$ be a curve in $\mathbb{P}^2$, and $x$ be a point of $\mathbb{P}^2$.

**Lemma (c)**

Suppose that no line containing $x$ is a component of the curve $C$. Then for a sufficiently general line $L_x$ passing through $x$, there is an epimorphism

$$\pi_1\left(L_x \setminus C\right) \longrightarrow \pi_1\left(\mathbb{P}^2 \setminus C\right)$$

induced by the inclusion.

A plane curve may contain a line through $x$ without being a cone, but such a curve cannot be a general hyperplane section through $x$. 
Part 3. Geography of singularities
We use our Lefschetz theorem to justify

**Theorem (B)**

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n-1)$-th sectional Milnor number of $V(h)$ at one of its points $x$. Then

$$\deg(\text{grad}(h)) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex $x$. 
Proof of Theorem (B) when $n \geq 3$.

We know that

$$\chi(D(h)) = (-1)^n \deg(\text{grad}(h)) + \sum_{i=0}^{n-1} (-1)^i (d - 1)^i.$$  

If $V(h)$ is not a cone with the apex $x$, choose a general hyperplane $H_x$ containing $x$ so that

(i) $V(h) \cap H_x$ is smooth outside $x$, and

(ii) the Milnor number of $V(h) \cap H_x$ at $x$ is the sectional Milnor number $\mu^{(n-1)}$. 

□
Proof of Theorem (B) when $n \geq 3$.

We know that

$$\chi(D(h)) = (-1)^n \deg(\text{grad}(h)) + \sum_{i=0}^{n-1} (-1)^i (d - 1)^i.$$

If $V(h)$ is not a cone with the apex $x$, choose a general hyperplane $H_x$ containing $x$ so that

(i) $V(h) \cap H_x$ is smooth outside $x$, and

(ii) the Milnor number of $V(h) \cap H_x$ at $x$ is the sectional Milnor number $\mu^{(n-1)}$.

Then

$$\text{rank } H_n\left(D(h), D(h) \cap H_x\right) = (-1)^n \left(\chi(D(h)) - \chi(D(h) \cap H_x)\right)$$

$$= \deg(\text{grad}(h)) - \mu^{(n-1)} \geq 0.$$
Therefore, if $\deg(\text{grad}(\mathcal{h})) = 1$, then $\mu^{(n-1)} = 1$ at all the singular points.
Therefore, if $\deg(\nabla h) = 1$, then $\mu^{(n-1)} = 1$ at all the singular points.

**Lemma (d)**

Let $(V, 0)$ be the germ of an isolated hypersurface singularity at the origin of $\mathbb{C}^n$. If $\mu^{(n-1)}$ of the germ is equal to 1, then the singularity is of type $A_k$ for some $k \geq 1$.

(The conjecture of DP follows.)
In conclusion, we have

**Theorem**

There is a forbidden value for the total Milnor number at the ‘top’, except for quadric hypersurfaces and cubic plane curves.
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**Theorem**

*There is a forbidden value for the total Milnor number at the ‘top’, except for quadric hypersurfaces and cubic plane curves.*

I believe

**Conjecture**

*This forbidden region is large if \( n \) and \( d \) are large. More precisely, for any positive integer \( k \), there is no projective hypersurface of polar degree \( k \) with only isolated singular points, for sufficiently large \( n \) and \( d \).*