TORIC ARRANGEMENTS

Emanuele Delucchi
(joint with Giacomo d’Antonio)
Universität Bremen

AMS Joint international meeting
Alba Iulia
June 28., 2013.
This talk will offer some snapshots of


A complexified toric arrangement is a set

\[ \mathcal{A} = \{(\chi_i, a_i)\}_{i=1,...,n} \subseteq \mathbb{Z}^d \times S^1. \]

With

\[ \mathbb{Z}^d \cong \text{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*), \]

\[ S^1 = \{z \in \mathbb{C} : |z| = 1\}, \]

define

\[ K_i = \chi_i^{-1}(a_i) \subseteq (\mathbb{C}^*)^d. \]
Toric arrangements

A complexified toric arrangement is a set

\[ \mathcal{A} = \{(\chi_i, a_i)\}_{i=1,...,n} \subseteq \mathbb{Z}^d \times S^1. \]

With

\[ \mathbb{Z}^d \simeq \text{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*), \]

\[ S^1 = \{ z \in \mathbb{C} : |z| = 1 \}, \]

define

\[ K_i = \chi_i^{-1}(a_i) \subseteq (\mathbb{C}^*)^d. \]

Layers of \( \mathcal{A} \): conn. comp. of intersection of some of the \( K_i \).

\( C(\mathcal{A}) := \) poset of layers ordered by reverse inclusion.
The arrangement $\mathcal{A}$ lifts to an hyperplane arrangement $\mathcal{A}^\uparrow$. 
The arrangement $\mathcal{A}$ lifts to an hyperplane arrangement $\mathcal{A}^\uparrow$.

The face poset $\mathcal{F}(\mathcal{A}^\uparrow)$ carries an action of $\mathbb{Z}^d$ “by translations”.

**FACE STRUCTURE**
The arrangement $\mathcal{A}$ lifts to an hyperplane arrangement $\mathcal{A}^\uparrow$.

The face poset $\mathcal{F}(\mathcal{A}^\uparrow)$ carries an action of $\mathbb{Z}^d$ “by translations”.

The face category of $\mathcal{A}$ is

$$\mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{A}^\uparrow)/\mathbb{Z}^d,$$

an acyclic category.
The arrangement $\mathcal{A}$ lifts to an hyperplane arrangement $\mathcal{A}^\uparrow$.

The face poset $\mathcal{F}(\mathcal{A}^\uparrow)$ carries an action of $\mathbb{Z}^d$ “by translations”.

The face category of $\mathcal{A}$ is

$$\mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{A}^\uparrow)/\mathbb{Z}^d,$$

an acyclic category.
is a central arrangement in $\mathbb{R}^d$, consisting of the translate at the origin of a lift of each $K_i$. 

$\mathcal{A}_0 = \{H_1, \ldots, H_n\}$
Given $Y \in \mathcal{C}(\mathcal{A})$ let

$$\mathcal{A}[Y] = \{ H_i \in \mathcal{A}_0 : Y \subseteq K_i \}.$$
Given $Y \in \mathcal{C}(\mathcal{A})$ let

$$\mathcal{A}[Y] = \{ H_i \in \mathcal{A}_0 : Y \subseteq K_i \}.$$ 

For $i = 1, \ldots, d$, let

$$\mathcal{N}_i := \{(Y, N) \mid Y \in \mathcal{C}(\mathcal{A}), N \in \text{nbc}(\mathcal{A}[Y]), |N| = \text{rk} \mathcal{A}[Y] = i\}.$$
We consider

$$M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i.$$
We consider

\[ M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i. \]

▶ [Looijenga ‘98; De Concini, Procesi ‘05] The Poincaré polynomial of \( M(\mathcal{A}) \) can be computed in terms of \( \mathcal{C}(\mathcal{A}) \).

▶ [dD ‘11] Presentation of \( \pi_1(M(\mathcal{A})) \) in terms of \( \mathcal{F}(\mathcal{A}) \).
The Poincaré polynomial of $M(\mathcal{A})$ is

$$P(M(\mathcal{A}), t) = \sum_{j=1}^{d} |N_j|(1+t)^{d-j} t^j$$

Moreover, when $\mathcal{A}$ is unimodular the multiplicative structure of $H^*(M(\mathcal{A}), \mathbb{C})$ is computed.
The Poincaré polynomial of $M(\mathcal{A})$ is

$$P(M(\mathcal{A}), t) = \sum_{j=1}^{d} |\mathcal{N}_j|(1+t)^{d-j} t^j$$

Moreover, when $\mathcal{A}$ is unimodular the multiplicative structure of $H^*(M(\mathcal{A}), \mathbb{C})$ is computed.

► Is there torsion in $H^*(M(\mathcal{A}), \mathbb{Z})$?

► What is the multiplicative structure of $H^*(M(\mathcal{A}), \mathbb{Z})$?

► When is $M(\mathcal{A})$ a $K(\pi, 1)$?

► Can the category $\mathcal{F}(\mathcal{A})$ be defined axiomatically?
Poincaré Polynomial

[De Concini and Procesi, ’05]

The Poincaré polynomial of \( M(\mathcal{A}) \) is

\[
P(M(\mathcal{A}), t) = \sum_{j=1}^{d} |N_j|(1+t)^{d-j} t^j
\]

Moreover, when \( \mathcal{A} \) is unimodular the multiplicative structure of \( H^*(M(\mathcal{A}), \mathbb{C}) \) is computed.

- Is there torsion in \( H^*(M(\mathcal{A}), \mathbb{Z})? \) [Today]
- What is the multiplicative structure of \( H^*(M(\mathcal{A}), \mathbb{Z})? \) [Ongoing project w. F. Callegaro]
- When is \( M(\mathcal{A}) \) a \( K(\pi, 1) \)?
- Can the category \( \mathcal{F}(\mathcal{A}) \) be defined axiomatically?
We consider

\[ M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i. \]
We consider

\[ M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i. \]

The action on \( \mathcal{A} \) extends to a cellular action on \( \text{Sal}(\mathcal{A}) \).

The Salvetti category of \( \mathcal{A} \) is the acyclic category

\[ \text{Sal}(\mathcal{A}) := \text{Sal}(\mathcal{A})/\mathbb{Z}^d. \]

Theorem [Moci and Settepanella ‘11, dD ‘11]. \( \text{Sal}(\mathcal{A}) \) can be defined in terms of \( \mathcal{F}(\mathcal{A}) \), and we have a homotopy equivalence

\[ \Delta(\text{Sal}(\mathcal{A})) \cong M(\mathcal{A}). \]
We consider

\[ M(\mathcal{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i. \]

The diagram of acyclic categories

\[ \mathcal{D} : \mathcal{F}(\mathcal{A}) \to \text{AC} \]

\[ F \mapsto \text{Sal}(\mathcal{A}[|F|]), \]

with inclusions as morphisms, is “geometric”.

**Theorem [dD ’12].**

\[ \text{colim} \mathcal{D} \simeq \text{Sal}(\mathcal{A}) \]
**Order on Chambers**

**Definition.** Let $C_1$, $C_2$ be chambers of $\mathcal{B}$, $F$ any face. 
$S(C_1, C_2) \subseteq \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$,

Fix a chamber $B$.

The partial order

$C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**Order on Chambers**

**Definition.** Let $C_1, C_2$ be chambers of $\mathcal{B}$, $F$ any face. $S(C_1, C_2) \subseteq \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order

$$C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**Definition.** Let $C_1, C_2$ be chambers of $\mathcal{B}$, $F$ any face. $S(C_1, C_2) \subseteq \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order

$$C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**ORDER ON CHAMBERS**

**Definition.** Let $C_1, C_2$ be chambers of $\mathcal{B}$, $F$ any face. $S(C_1, C_2) \subset \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order

$$C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 

![Diagram showing chambers and hyperplanes](image)
**Definition.** Let $C_1$, $C_2$ be chambers of $\mathcal{B}$, $F$ any face. 
$S(C_1, C_2) \subseteq \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order

$C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**Definition.** Let $C_1, C_2$ be chambers of $\mathcal{B}$, $F$ any face. $S(C_1, C_2) \subset \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order $C_1 \leq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**Order on Chambers**

**Definition.** Let $C_1, C_2$ be chambers of $\mathcal{B}$, $F$ any face.

$S(C_1, C_2) \subseteq \mathcal{B}$: the set of hyperplanes separating $C_1$ from $C_2$.

Fix a chamber $B$.

The partial order

$$C_1 \preceq_B C_2 \iff S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathcal{B})$ based at $B$.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathcal{B})$. 
**Definition: $X_C$**

For every chamber $C$ there a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ s.t. the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
**Definition:** $X_C$

For every chamber $C$ there a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ s.t. the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
**Definition: $X_C$**

For every chamber $C$ there a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ s.t. the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
**DEFINITION: $X_C$**

For every chamber $C$ there a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ s.t. the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
DEFINITION: $X_C$

For every chamber $C$ there a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ s.t. the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
**DEFINITION: $X_C$**

For every chamber $C$ there is a unique minimal $X_C \in \mathcal{L}(\mathcal{B})$ such that the set

$$\{H \in \mathcal{B} : H \supseteq X_C\}$$

separates $C$ from the previous.
Let $\mathcal{B}$ be a central arrangement of real hyperplanes, fix $B \in \mathcal{P}(\mathcal{B})$.

**Theorem [D. ‘08].** The order preserving map

$$\phi : \mathcal{P}_B(\mathcal{B}) \to \mathcal{L}(\mathcal{B}), \ C \mapsto \chi_C$$

satisfies

$$|\phi^{-1}(Y)| = |\{N \in \text{nbc} B | \cap N = Y\}|$$
$X_C$: TWO APPLICATIONS

Let $\mathcal{B}$ be a central arrangement of real hyperplanes, fix $B \in \mathcal{P}(\mathcal{B})$.

**Theorem [D. ‘08].** The order preserving map

$$\phi : \mathcal{P}_B(\mathcal{B}) \to \mathcal{L}(\mathcal{B}), \ C \mapsto X_C$$

satisfies

$$|\phi^{-1}(Y)| = |\{ N \in \text{nbc}(\mathcal{B}) \mid \cap N = Y \}|$$

**Theorem [D. ‘08].** There is an order preserving map

$$\text{Sal}(\mathcal{B}) \to \mathcal{P}_B(\mathcal{B})$$

such that for the preimage $N_C$ of every $C \in \mathcal{P}_B(\mathcal{B})$ we have a poset isomorphism

$$N_C \simeq \mathcal{F}(\mathcal{B}^{X_C})^{op}$$
COMBINATORIAL BOOKKEEPING II

Fix
• $B \in \mathcal{P}(\mathcal{A}_0)$ and
• a lin. ext. of $\mathcal{P}_B(\mathcal{A}_0)$.

For all $Y \in C(\mathcal{A})$, we have
• $B_Y \in \mathcal{P}(\mathcal{A}[Y])$ with $B \subseteq B_Y$
• a lin. ext. of $\mathcal{P}_{B_Y}(\mathcal{A}[Y])$. 
COMBINATORIAL BOOKKEEPING II

Fix
• \( B \in \mathcal{P}(\mathcal{A}_0) \) and
• a lin. ext. of \( \mathcal{P}_B(\mathcal{A}_0) \).

For all \( Y \in \mathcal{C}(\mathcal{A}) \), we have
• \( B_Y \in \mathcal{P}(\mathcal{A}[Y]) \) with \( B \subseteq B_Y \)
• a lin. ext. of \( \mathcal{P}_{B_Y}(\mathcal{A}[Y]) \).

For every \( i = 0, \ldots, d \) define
\[
\mathcal{Y}_i := \{(Y, C) \mid Y \in \mathcal{C}(\mathcal{A}), C \in \mathcal{P}_{B_Y}(\mathcal{A}[Y]), X_C = \max \mathcal{L}(\mathcal{A}[Y])\}.
\]

Then,
\[
|\mathcal{Y}_i| = |\mathcal{N}_i|.
\]
Fix
- $B \in \mathcal{P}(\mathcal{A}_0)$ and
- a lin. ext. of $\mathcal{P}_B(\mathcal{A}_0)$.

For all $Y \in \mathcal{C}(\mathcal{A})$, we have
- $B_Y \in \mathcal{P}(\mathcal{A}[Y])$ with $B \subseteq B_Y$
- a lin. ext. of $\mathcal{P}_{B_Y}(\mathcal{A}[Y])$.

Let $\mathcal{Y} := \bigcup_i \mathcal{Y}_i$. For every $(Y, C) \in \mathcal{Y}$ define a subdiagram of $\mathcal{D}$

$$\mathcal{N}_{(Y, C)} : \mathcal{F}(\mathcal{A}^Y) \to \mathcal{AC}$$

$$F \mapsto \mathcal{N}_C(\mathcal{A}[|F|]).$$

**Theorem [dD ‘12].** This diagram is geometric, and

$$\text{colim} \mathcal{N}_{(Y, C)} \simeq \mathcal{F}(\mathcal{A}^Y)$$
Fix
• $B \in \mathcal{P}(A_0)$ and
• a lin. ext. of $\mathcal{P}_B(A_0)$.
Choose total order on $Y$ such that the natural map

$$Y \to \mathcal{P}_B(A_0)$$

is order preserving.
Fix

• $B \in \mathcal{P}(\mathcal{A}_0)$ and
• a lin. ext. of $\mathcal{P}_B(\mathcal{A}_0)$.

Choose total order on $Y$ such that the natural map

$$Y \rightarrow \mathcal{P}_B(\mathcal{A}_0)$$

is order preserving.

**Theorem [dD ‘12].** There is a functor

$$\Phi : \text{colim } \mathcal{D} \rightarrow Y$$

with

$$\Phi^{-1}(Y, C) = \text{colim } \mathcal{N}_{(Y, C)}$$
STRAITIFICATION

Fix
• $B \in \mathcal{P}(\mathcal{A}_0)$ and
• a lin. ext. of $\mathcal{P}_B(\mathcal{A}_0)$.
Choose total order on $\mathcal{Y}$ such that the natural map

$$\mathcal{Y} \rightarrow \mathcal{P}_B(\mathcal{A}_0)$$

is order preserving.

We obtain a functor

$$\Phi : \text{Sal}(\mathcal{A}) \rightarrow \mathcal{Y}$$

with

$$\Phi^{-1}(\mathcal{Y}, C) = F(\mathcal{A}^\mathcal{Y}),$$

which allows us to turn to Discrete Morse Theory.
Here is a regular CW complex with its poset of cells:
ELEMENTARY COLLAPSES...

... are homotopy equivalences.

Cells:
Elementary collapses...

... are homotopy equivalences.

Cells:
ELEMENTARY COLLAPSES...

... are homotopy equivalences.
Elementary collapses...

Cells:

... are homotopy equivalences.
ELEMENTARY COLLAPSES...

... are homotopy equivalences.
Elementary collapses...

... are homotopy equivalences.
ELEMENTARY COLLAPSES...

Cells:

... are homotopy equivalences.
Elementary collapses...

Cells:

... are homotopy equivalences.
Elementary collapses...

Cells:

... are homotopy equivalences.
ACYCLIC MATCHINGS

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.
ACYCLIC MATCHINGS

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.

**Question:** Does any matchings encode such a sequence?
ACYCLIC MATCHINGS

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.

Question: Does any matchings encode such a sequence?
Answer: No. Only (and exactly) those without “cycles” like

Acyclic matchings ↔ discrete Morse functions.
DMT FORACYCLIC CATEGORIES

Meta-Theorem [dD ‘12]. Discrete Morse Theory generalizes successfully to nerves of acyclic categories.

In particular, we have

- A notion of ‘acyclic matching’
- A corresponding ‘main theorem’
- A corresponding ‘Patchwork Lemma’:
Meta-Theorem [dD ‘12]. Discrete Morse Theory generalizes successfully to nerves of acyclic categories.

In particular, we have

- A notion of ‘acyclic matching’
- A corresponding ‘main theorem’
- A corresponding ‘Patchwork Lemma’:

Let \( \varphi : \mathcal{A} \to \mathcal{B} \) be a functor of acyclic categories. For \( b \in \text{Ob}(\mathcal{B}) \) let \( M_b \) be an acyclic matching of the preimage \( \varphi^{-1}(b) \). Then, the union \( M := \bigcup_b M_b \) is an acyclic matching of \( \mathcal{A} \).
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
PERFECT MATCHINGS FOR REAL TORIC ARRANGEMENTS
Perfect matchings for real toric arrangements

Posets of interior cells of constructible complexes admit acyclic matchings with only one critical cell. [Benedetti '10]
Lemma [dD ‘12]. The category $\mathcal{F}(\mathcal{A})$ admits an acyclic matching with $2^d$ critical cells in total.

Posets of interior cells of constructible complexes admit acyclic matchings with only one critical cell. [Benedetti ’10]
FIT FOR MINIMALITY

Theorem [dD ‘12]. Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.
FIT FOR MINIMALITY

Theorem [dD ‘12]. Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.

Proof.

- For every $y = (Y, C) \in \mathcal{Y}$ we have a stratum
  $\mathcal{N}_y \simeq \mathcal{F}(\mathcal{A} \cap Y)^{op}$.
FIT FOR MINIMALITY

Theorem [dD ‘12]. Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.

Proof.

- For every $y = (Y, C) \in \mathcal{Y}$ we have a stratum
  
  $N_y \simeq F(\mathcal{A} \cap Y)^{op}$.

- It admits an acyclic matching with $2^{\dim(Y)}$ critical cells.

Corollary. All cohomology modules $H^k(M(\mathcal{A}), \mathbb{Z})$ are torsion free.
**FIT FOR MINIMALITY**

**Theorem [dD ‘12].** Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.

**Proof.**

- For every $y = (Y, C) \in \mathcal{Y}$ we have a stratum $\mathcal{N}_y \simeq \mathcal{F}(\mathcal{A} \cap Y)^{op}$.

- It admits an acyclic matching with $2^{\dim(Y)}$ critical cells.

- The functor $\Phi : \text{Sal}(\mathcal{A}) \to \mathcal{Y}$, has $\Phi^{-1}(y) = \mathcal{N}_y$. 

**Corollary.** All cohomology modules $H^k(M(\mathcal{A}), \mathbb{Z})$ are torsion free.
FIT FOR MINIMALITY

Theorem [dD ‘12]. Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.

Proof.

- For every $y = (Y, C) \in \mathcal{Y}$ we have a stratum $\mathcal{N}_y \simeq \mathcal{F}(\mathcal{A} \cap Y)^{op}$.

- It admits an acyclic matching with $2^{\dim(Y)}$ critical cells.

- The functor $\Phi : \text{Sal}(\mathcal{A}) \to \mathcal{Y}$, has $\Phi^{-1}(y) = \mathcal{N}_y$.

Patchwork Lemma: number of critical cells over all of $\text{Sal}(\mathcal{A})$:

$$\sum_{(Y, C) \in \mathcal{Y}} 2^{\dim Y} = \sum_{(Y, N) \in \mathcal{N}} 2^{\dim Y} = \sum_{j=1}^{d} |\mathcal{N}_j|(1 + 1)^{d-j}1^j = P_{M(\mathcal{A})}(1).$$

Corollary. All cohomology modules $H^k(M(A), \mathbb{Z})$ are torsion free.
FIT FOR MINIMALITY

Theorem [dD ‘12]. Let $\mathcal{A}$ be a complexified toric arrangement. Then $M(\mathcal{A})$ has the homotopy type of a minimal CW-complex.

Proof.

- For every $y = (Y, C) \in \mathcal{Y}$ we have a stratum $\mathcal{N}_y \simeq \mathcal{F}(\mathcal{A} \cap Y)^{op}$.

- It admits an acyclic matching with $2^{\dim(Y)}$ critical cells.

- The functor $\Phi : Sal(\mathcal{A}) \to \mathcal{Y}$, has $\Phi^{-1}(y) = \mathcal{N}_y$.

**Patchwork Lemma:** number of critical cells over all of $Sal(\mathcal{A})$:

$$\sum_{(Y, C) \in \mathcal{Y}} 2^{\dim Y} = \sum_{(Y, N) \in \mathcal{N}} 2^{\dim Y} = \sum_{j=1}^{d} |\mathcal{N}_j|(1 + 1)^{d-j}j = P_{M(\mathcal{A})}(1).$$

**Corollary.** All cohomology modules $H^k(M(\mathcal{A}), \mathbb{Z})$ are torsion free.