IEEE Workshop on Mathematical Methods in Biomedical Image Analysis, San Francisco, June 21–22, 1996.

A Computationally Efficient Shape Analysis via Level Sets

Z. Sibel Gőktepe Tari
Mechanical and Industrial Engineering Department
Northeastern University, Boston, MA 02115

Jayant Shah
Mathematics Department, Northeastern University, Boston, MA 02115

Homer Pien
Draper Laboratory, Cambridge, MA 02139

Abstract[1]

In recent years, curve evolution has been applied to smoothing of shapes and shape analysis with considerable success, especially in biomedical image analysis. The multiscale analysis provides information regarding parts of shapes, their axes or centers and shape skeletons. In this paper, we show that the level sets of an edge-strength function provide essentially the same shape analysis as provided by curve evolution. The new method has several advantages over the method of curve evolution. Since the governing equation is linear, the implementation is simpler and faster. The same equation applies to problems of higher dimension. An important advantage is that unlike the method of curve evolution, the new method is applicable to shapes which may have junctions such as triple points. The edge-strength may be calculated from raw images without first extracting the shape outline. Thus the method can be applied to raw images. The method provides a way to approach the segmentation problem and shape analysis within a common integrated framework.

1. Introduction

In recent years, curve evolution has been applied to smoothing of shapes [1,5,13] and shape analysis [6,7,18] with considerable success. The underlying principle is the evolution of a simple closed curve whose points move in the direction of the normal with prescribed velocity. Kimia, Tannenbaum and Zucker [5] proposed evolution of the curve by letting its points move with velocity consisting of two components: a smoothing component proportional to curvature and a constant component corresponding to morphology. Depending on the sign of the constant component of the velocity, the curve can expand (thus joining two disjoint nearby shapes) or contract (thus separating a dumbbell shape into two separate blobs). The formulation involves one parameter which together with time provides a two-dimensional scale space, called "entropy" scale space of the shape [7]. Keeping track of how singularities develop and disappear as the curve evolves provides information regarding the geometry of the shape in terms of its parts and its skeleton [6,18].

The easiest way to implement curve evolution is by embedding the intial curve as a level curve in a surface and let all the level curves of the surface evolve simultaneously. The advantage is that changes in the topology of the curve are handled automatically, simplifying the data structure. The usual way to embed the curve is by means of the (signed) distance function so that the given curve is defined as the locus of the zero-crossings of the distance function. As the surface evolves, it traces out a three-dimensional volume. Embedded in this volume are the surfaces traced out by the level curves of the initial surface. Numerical scheme of Osher and Sethian [12] may then be used to implement the evolution of the surface. This is how the method is normally used.

In this paper, a simpler and faster method is proposed to obtain essentially the same shape analysis as provided by the method of curve evolution. During the curve evolution, the time t may be thought of as a function over the plane by setting $t(x, y)$ = the time when the evolving curve passes through the point (x, y) if it does and equal to $-\infty$ if it does not, assuming that the evolving curve passes through any point in the plane at most once. Then, the level curves of the surface $t(x, y)$ describe the evolution of the initial curve. (Note however that in a computational framework like the one by Osher and Sethian, the surface $t(x, y)$ is realized as an embedded

[1] This research was partially supported under NIH Grant No. I-R01–NS34189–01.

surface in a three-dimensional manifold.) In place of the function $t(x,y)$, we propose to use an edge-strength function v whose level curves exhibit properties similar to those of the level curves of t. More specifically, we show that as the constant component of the velocity tends to infinity, the level curves of v obey approximately the same evolution equation as those of t. Function $t(x,y)$ may be thought of as a first order approximation of v. The function v depends on a parameter ρ so that v and ρ parametrize a two-dimensional scale space for the shape analogous to the entropy scale space of Kimia et al. The geometry of the surface defined by v contains information regarding the shape skeleton and the decomposition of the shape into parts.

The alternate framework of the edge-strength function proposed here has a significant computational advantage over curve evolution. First of all, v may be calculated by solving a *linear* diffusion equation which is easy to implement by standard finite difference methods. In contrast, the equation of curve evolution is nonlinear and because the evolving surface develops shocks, standard finite difference methods (for example, central differences) cannot be used. One must use a shock-capturing scheme such as the one proposed by Osher and Sethian in which the direction of the finite difference depends on the direction in which the shock is developing. Secondly, the fact that the surface $t(x,y)$ is realized as the locus of zero-crossings of a three-dimensional manifold adds another layer of computational complexity to the task of locating the singularities of its level curves [18]. Lastly, the new formulation does not require the initial computation of the distance transform.

Another key advantage of the new framework is that it removes the severe restriction imposed by the method of curve evolution on the initial curve, namely, that it must be a simple closed curve. Consequently, the new method may be applied to a collection of curves which need not be disjoint or closed. In particular, the method permits analysis of shapes involving Y-junctions such as the line-drawing of a solid cube and it can be applied to incomplete shape outlines consisting of a collection of disconnected pieces.

Finally, the method proposed here can be applied to raw images and provides an integrated approach to the segmentation problem and shape analysis. A central assumption throughout the discussion above is that the shape outline is already extracted in the form of a closed curve from the raw image. This of course is not an easy task if the image is noisy. A number of recent papers [3,4,8,9,14,15,19] has been devoted to shape recovery from raw images by the method of curve evolution. (See [16] for a discussion and a generalization of these methods.) However, one may calculate the edge-strength function corresponding to shape boundaries directly from the raw image without first segmenting it or resorting to curve evolution for shape recovery. The level curves of this edge-strength function may then be analyzed for shape information. The prime example of such an edge-strength function is the one constructed by Ambrosio and Tortorelli [2] for approximating a segmentation functional.

This paper is organized as follows. In §2, we review curve evolution. The edge–strength function and interpretation of its level curves as a scale space for shapes are described in §3 with illustrative examples. In §4, we define shape skeleton and shape decomposition and show the results for several test cases. In §5, we briefly describe the Ambrosio-Tortorelli approximation of a segmentation functional and apply it to analyze an MRI image.

2. A Review of Curve Evolution

Let Γ be a simple closed curve in the plane. Let $C(p,t): I \times [0,\infty) \rightarrow \mathbf{R^2}$ be the evolving family of curves where I is the unit interval and t denotes time. (In practice, $\mathbf{R^2}$ is replaced by a bounded domain such as a square.) We require that $C(0,t) = C(1,t)$ for all values of t and the image of $C(p,0)$ coincides with Γ. Let N denote the inward normal and κ the curvature which is defined such that it is positive when Γ is a circle. Then the evolution of the curve moving inward with velocity = $\alpha + \kappa$ where α is a constant is governed by the equation

$$\frac{\partial C}{\partial t} = [\alpha + \kappa]N \tag{1}$$

In order to implement the evolution of Γ, assume that Γ is embedded in a surface $f_0 : \mathbf{R^2} \rightarrow \mathbf{R}$ as the locus of its zero-crossings. For the sake of definiteness, assume that $f_0 < 0$ inside Γ. Let $f(t,x,y)$ denote the evolving surface such that $f(0,x,y) = f_0(x,y)$. Then, in order to let all the level curves of f_0 evolve simultaneously, let f evolve according to the equation (see [6])

$$\frac{\partial f}{\partial t} = -[\alpha + \ curv(f)]||\nabla f|| \tag{2}$$

where

$$curv(f) = \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{\left(f_x^2 + f_y^2\right)^{3/2}} \tag{3}$$

$curv(f)$ is the curvature of the level curves of f. The locus of zeros of $f(x,y,t)$ describes the evolution of Γ.

3. Edge-Strength Function and Scale Space of a Shape

Again, let Γ be a curve in the plane, not necessarily a simple closed curve. We consider the following functional introduced by Ambrosio and Tortorelli [2]:

$$\Lambda_\rho(v) = \frac{1}{2}\int\int\left\{\rho\|\nabla v\|^2 + \frac{v^2}{\rho}\right\}dxdy \tag{4}$$

subject to the boundary condition $v = 1$ along Γ. The functional is designed so that as the parameter $\rho \to 0$, $\min \Lambda_\rho(v)$ tends to the length of Γ. The equation of steepest gradient descent for the functional is the linear diffusion equation:

$$\frac{\partial v}{\partial \tau} = \nabla^2 v - \frac{v}{\rho^2} \tag{5}$$

which is easier to implement than equation (2).

Let v denote the unique minimizer of the functional Λ_ρ. Then, v varies between 0 and 1 and for sufficiently small values of ρ,

$$v \approx e^{-\frac{d}{\rho}} \tag{6}$$

where d is the (unsigned) distance from Γ. Thus, v may be thought as a blurred version of Γ and ρ as the blurring radius.

In order to relate v to curve evolution, assume that Γ is a simple, closed curve. As shown in Appendix (3) of [11], inside Γ,

$$v(x,y) = -\rho\left(1 + \frac{\rho\kappa(x,y)}{2}\right)\frac{\partial v}{\partial n}(x,y) + O(\rho^3) \tag{7}$$

where $\kappa(x,y)$ is the curvature of the level curve of v passing through the point (x,y) and n is the direction of the inward normal. Therefore, if we imagine moving from a level curve to a level curve along the normals, then for small values of ρ, a change of Δv in level requires movement

$$\Delta r \approx -\frac{\rho}{v}\left(1 + \frac{\rho\kappa}{2}\right)(\Delta v) \tag{8}$$

where r denotes the arc length along the gradient lines of v, (the positive direction being the direction of the inward normals). Let $\Delta t = -\frac{\rho^2 \Delta v}{2v}$. (Notice that v is a decreasing function in the direction of the inward normal so that Δv is negative. Functions v and t have the same set of level curves because t is a monotonic function of v.) Passing to the infinitesimals, we get the velocity:

$$\frac{dr}{dt} \approx \frac{2}{\rho} + \kappa \tag{9}$$

in agreement with equation (1).

A similar argument shows that for moving outward from Γ, the velocity is given by the formula

$$\frac{dr}{dt} \approx -\frac{2}{\rho} + \kappa \tag{10}$$

That is, the sign of the morphology component of the velocity is reversed. Consequently, the totality of the level curves of v, outside and inside of Γ cover a range of positive and negative values of α in equation (2). However, the proposed method does not extend to the limiting cases, namely, the case of pure morphological evolution obtained by omitting from (1) the curvature term (that is, the limit as $\alpha \to \infty$) and the case of pure smoothing obtained by setting $\alpha = 0$. This is not a serious disadvantage. On one hand, pure morphological evolution is very sensitive to noise which makes inclusion of the curvature term essential; on the other hand, pure smoothing shrinks every shape to a single "round" point and it seems perceptually unnatural to reduce shapes which deviate a great deal from being a circle – shapes such as dumbbells and spirals – to a single point.

Figure 1 shows the scale space obtained by our method for three different shapes. (All the figures in this paper except the figure of a doll in Figure 2 were represented on a 256×256 lattice. The doll figure was on a 128×128 lattice.) The two top rows show the level curves for a duck with ρ equal to 4 and 32. The stronger smoothing effect due to the higher value of ρ is clearly visible. The bottom row of the figure depicts the level curves for the line drawing of a solid cube and a pair of pliers, illustrating the applicablity of the method when the initial shape has junctions and self-intersections.

Figure 2 shows how various shapes evolve and decompose. We used doll and hand image for the purpose of comparison with similar description by the standard curve evolution [7]. Qualitatively, the two descriptions are indistinguishable.

4. Shape Skeleton and Shape Decomposition

In a purely morpholgical evolution, singularities develop as corners and self-intersections form. The locus of these singularities is called the skeleton of the shape. When smoothing is introduced, self-intersections still may develop (due to thinning of narrow necks), but the corners are rounded out. Therefore, when smoothing is present, points of maximum curvature serve as a substitute for corners. In the case of the standard curve evolution, the points of maximum curvature correspond to the points of maximum velocity. Even though this is only approximately true in our formulation, it provides an alternative way of defining shape skeleton, namely, by determining points where $|\nabla v|$ is minimum. (Note that, from Equations (7), (8) and (9), $\frac{dr}{dt} \cdot \left|\frac{\partial v}{\partial n}\right| = \frac{2v}{\rho^2}$ and so maximum velocity corresponds to minimum gradient.) We prefer this alternative because computation of curvature involves second derivatives of v and hence it is more sensitive to noise than $|\nabla v|$.

Let K^+ denote the closure of the set of zero-crossings of $\frac{d|\nabla v|}{ds}$ where $\frac{d^2|\nabla v|}{ds^2}$ is positive. Here, s denotes the arc-length along the level curves.

$$\begin{aligned} \frac{d|\nabla v|}{ds} &= v_{\eta\xi} \\ \frac{d^2|\nabla v|}{ds^2} &= v_{\eta\xi\xi} + \frac{v_{\xi\xi}(v_{\xi\xi} - v_{\eta\eta})}{|\nabla v|} \end{aligned} \tag{11}$$

where

$$\begin{aligned} v_{\eta\xi} &= \frac{\left\{ \left(v_y^2 - v_x^2\right) v_{xy} - v_x v_y \left(v_{yy} - v_{xx}\right) \right\}}{|\nabla v|^2} \\ v_{\xi\xi} &= \frac{\left\{ v_y^2 v_{xx} - 2 v_x v_y v_{xy} + v_x^2 v_{yy} \right\}}{|\nabla v|^2} \\ v_{\eta\eta} &= \frac{\left\{ v_x^2 v_{xx} + 2 v_x v_y v_{xy} + v_y^2 v_{yy} \right\}}{|\nabla v|^2} \\ v_{\eta\xi\xi} &= \frac{1}{|\nabla v|^3} \{ v_x v_y^2 v_{xxx} + v_y \left(v_y^2 - 2v_x^2\right) v_{xxy} \\ &\qquad + v_x \left(v_x^2 - 2v_y^2\right) v_{xyy} + v_x^2 v_y v_{yyy} \} \end{aligned} \tag{12}$$

(Interestingly, $\frac{1}{|\nabla v|} \frac{d|\nabla v|}{ds}$ is the curvature of the gradient lines.)

Points where the (signed) curvature attains its minimum are also of interest because such points indicate indentations and necks. Therefore,

let K^- denote the closure of the set of zero-crossings of $\frac{d|\nabla v|}{ds}$ where $\frac{d^2|\nabla v|}{ds^2}$ is negative.

Let $K = K^+ \cup K^-$.

Let S denote the set of points where $|\nabla v| = 0$.

The direction of evolution at each point of K is the direction of decreasing v. For a perfect circle, K^+ and K^- are empty and S consists of a single point which is the unique minimum of v. A simple closed curve may be thought of as a deformation of a circle by means of protrusions and indentations. As it evolves towards a more circular shape, K^+ tracks evolution of its protrusions while K^- tracks evolution of its indentations. During the evolution, a protrusion might merge with an indentation, ($d|\nabla v|/ds = d^2|\nabla v|/ds^2 = 0$), joining a branch of K^+ with a branch K^- and terminating both the branches. Of course, more complicated merges between the branches of K^+ and the branches of K^- are also possible and in such a case, a new branch might start from the junction. It is also possible that a branch might bifurcate. This typically happens just before a thinning neck breaks up. As two indentations begin to evolve towards each other, each one gives rise to a branch of K^-. However, as the indentations approach a break-point, they interact and slow down the rate of decay of v. Each branch of K^- splits into three branches, the middle branch belongs to K^+ and continues towards the break–point while the other two belong to K^-. If a branch is not terminated at a junction, then it will terminate at a point in S. If the point is a minimum point of v, then the evolution has come to rest at that point and it is appropriate to call such a point the center of a part. If the point is not a minimum, then it may signify a change in the topology of the evolving curve, that is, break-up of the shape due to a thinning neck. If the point signifies a change in the topology of the shape, we will call it a saddle point. There are at least two branches of K^+ leaving such a point.

In differential geometric terms, a point in S is either elliptic, hyperbolic and parabolic depending on whether the determinant of the hessian of v, $v_{xx}v_{yy} - v_{xy}^2$, is positive, negative or zero respectively. An elliptic point is always a center and a hyperbolic point is always a saddle point. Parabolic points are more troublesome to classify because of their global nature. We have to analyze the branches of K meeting the parabolic point (or the connected component of S containing the parabolic point in case it is not an isolated parabolic point). If we can find at least two branches of K^+ leaving from the connected component of S containing a given parabolic point, then it is a saddle point. If all the branches of K meeting the connected component containing the parabolic point lead into it, it is a center.

Now we can define the skeleton of a shape.

Definition: The *skeleton* of a shape is the subset of $K^+ \cup S$ which excludes those branches of K^+ which flow into a connected component of S containing a saddle point.

The definition is designed so as exclude the branches of K^+ along which necks of the shape evolve towards a break.

We can also define the segmentation of a shape corresponding to its break-up into parts during evolution due to the presence of narrow necks.

Definition: The segmentation of a shape is the union of branches of K which flow into a saddle point.

Note that since a branch of K^+ may terminate at a junction with a branch of K^-, the skeleton need not be connected. In our description, the skeleton always extends to the boundary while in the purely morphological evolution, a branch starts only after a corner has formed. We could approximate the purely morphological skeleton by deleting from the skeleton defined above all points where the curvature is below a certain threshold. Note also that the definition of segmentation does not deal with protrusions. Significant protrusions such as the fingers of a hand have to be recovered as parts of the shape from the branches of the skeleton.

Although the constructions described above were motivated by the example of simple closed curves, they work equally well for more general curves. For example, if the letter C is drawn as an open curve, we can compute its skeleton. If the ends of the letter are sufficiently close, a segmentation line joins the two ends. If the same letter is drawn as a thick shape with a simple closed curve as its boundary, we recover the same information as above and also the medial axis. The point is that v should be computed over the whole plane and the skeleton and the segmentation really describe the complement of Γ. Whether the complement of Γ is connected or not is not relevant to the computation of the skeleton and the segmentation. This property is very useful in particular when the shape boundary is not fully specified. As long as important features of the shape boundary are specified, evolution fills in the gaps and the essential skeleton can be recovered. The missing portions of the boundary are recovered as branches of the segmentation.

This description can be readily translated into the language of the shock grammar of Siddiqi and Kimia [6,18]. The first order shocks are the branches of the skeleton not belonging to S. The second order shocks are the hyperbolic points. The third order shocks correspond to a line of parabolic points. The elliptic points are the fourth order shocks. The rules of the grammar and properties follow easily from the fact that v is monotonically decreasing along the branches of the skeleton and the fact the v is smooth.

In this paper, we have not addressed the issue of assigning a level of significance to each branch of the skeleton and the segmentation. One of the simplest criterion is the "time of extinction", measured in our case by the value of $(1-v)$ when the branch terminates. Another measure of significance could be based on the curvature of the level curves where they intersect the skeleton.

We explain our construction in detail by means of an example of two overlapping squares shown in Figure 3. The shape boundary is given in the top left and next to it are the level curves depicting the evolution of the shape. The shape skeleton (top right) is what one would expect. In the bottom row, we analyze the singularities. The leftmost figure shows the zero–crossings of the x and y derivatives of the edge-strength function v, their intresections corresponding to the singular points where the gradient vanishes. The three figures that follow show the surface corresponding to the edge-strength function in a neighborhood of these singular points. The one in the middle corresponds to the singularity at the neck and the surface at this point is hyperbolic, the other two correspond to the centers of the two squares and the surface there is elliptic.

Figure 4 shows the skeletons and segmentation for various shapes. We get nontrivial segmentation only in the case of the duck figure and the incomplete rectangle. The skeleton for the hand is qualitatively the same as in [18]. The duck is segmented across its neck. Examples of a pair of pliers and the outline of a cube illustrate the effectiveness of the method for complex shapes involving junctions. The incomplete rectangle was obtained by removing a 27 pixel long piece from the middle of each side of the complete rectangle. The skeleton consists of the skeleton of the complete rectangle and four new branches which may be interpreted as the "medial axes" of the gaps in the sides. The larger the value of ρ, the weaker these additional branches. The gaps themselves are filled in by segmentation curves.

In these examples, we didn't do any pruning.

5. Raw Images

Ambrosio and Tortorelli introduced the edge-strength function for obtaining an elliptic approximation of the following segmentation functional introduced in [10]:

$$\begin{aligned} E_{MS}(u,B) = & \alpha \int\int_{R\backslash B} \|\nabla u\|^2 dxdy \\ & + \beta \int\int_R (u-g)^2 dxdy + |B| \end{aligned} \tag{13}$$

where R is a connected, bounded, open subset of $\mathbf{R}^2$ (usually a rectangle), g is the feature intensity, B is a curve segmenting R, u is the smoothed image $\subset R\backslash B$, $|B|$ is the length of B and α, β are the weights. Let $\sigma = \sqrt{\alpha/\beta}$. Then, σ may be interpreted as the smoothing radius in $R\backslash B$. With σ fixed, the higher the value of α, the lower the penalty for B and hence, the more detailed the segmentation.

Ambrosio and Tortorelli [2] replace

$$|B| \text{ by } \frac{1}{2}\int\int_R \left\{\rho\|\nabla v\|^2 + \frac{v^2}{\rho}\right\} dxdy \tag{14}$$

and

$$\int\int_{R\backslash B} \|\nabla u\|^2 dxdy \text{ by } \int\int_R (1-v)^2 \|\nabla u\|^2 dxdy \tag{15}$$

The result is the following functional:

$$\begin{aligned} E_{AT}(u,v) = & \int\int_R \{\alpha(1-v)^2\|\nabla u\|^2 \\ & + \beta(u-g)^2 + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho}\} dxdy \end{aligned} \tag{16}$$

As $\rho \to 0$, v converges to 1 at points on B and to zero elsewhere. The corresponding gradient descent equations are:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (1-v)^2 \nabla u - \frac{\beta}{\alpha}(u-g) \\
\frac{\partial v}{\partial t} &= \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1-v)||\nabla u||^2 \\
\frac{\partial u}{\partial n}|_{\partial R} &= 0 \qquad \frac{\partial v}{\partial n}|_{\partial R} = 0
\end{aligned}
\tag{17}
$$

where ∂R denotes the boundary of R and n denotes the direction normal to ∂R. The solution of these equations gives us the edge-strength function v corresponding to the segmentation locus B without determining B itself.

Notice that equation for each variable is a diffusion equation which minimizes a convex quadratic functional in which the other variable is kept fixed:

(18) Keeping v fixed, the first equation minimizes

$$\int\int_R \left\{\alpha(1-v)^2||\nabla u||^2 + \beta(u-g)^2\right\}dxdy$$

Keeping u fixed, the second equation minimizes

$$\int\int_R \{||\nabla v||^2 + \frac{1+2\alpha\rho||\nabla u||^2}{\rho^2}\left(v - \frac{2\alpha\rho||\nabla u||^2}{1+2\alpha\rho||\nabla u||^2}\right)^2\}dxdy$$

Thus the edge strength function v is nothing but a non-linear smoothing of

$$\frac{2\alpha\rho||\nabla u||^2}{1+2\alpha\rho||\nabla u||^2} \tag{19}$$

where u is a simultaneous nonlinear smoothing of g.

Ideally, the edge-strength function v computed from a raw image by equations (17) should be constant along the object boundaries. However this almost never happens due to noise, differing levels of contrast along the object boundaries and the interaction between nearby edges. Therefore, the object boundaries are no longer defined by level curves of v. Typically, we should expect a level curve corresponding to a value of v near its maximum to consist of several connected components, each surrounding a high contrast portion of B. The situation is analogous to the earlier example of the incomplete rectangle where the shape boundary consists of several disconnected pieces. Note that even in the case of the functional (13), if α is not high enough, then B may not include portions of the object boundary where contrast is too low. The important point is that it is not essential to have the complete shape boundary to compute its essential skeleton. As the evolution progresses, the gaps between the pieces of the boundary are filled in. Thus we still recover an essentially correct skeleton. As the value of α is increased, the skeleton becomes more and more detailed.

In dealing with raw images, the necessity for assigning a level of significance to each branch of the skeleton or the segmentation and for pruning becomes important because of the presence of noise. Since smoothing of the image is minimal near the boundary where v is high, the skeleton tends to be noisier near the boundary. As mentioned before, we have not dealt with this issue in this paper. For the purpose of illustrations in this paper, instead of extracting the skeleton and pruning, we have pruned K^+ near the boundary by simply removing parts of K^+ which have values of v above a manually chosen threshold.

In order to illustrate application of the method to raw images, we present the example shown in Figure 5 and analyze the dark shape in the center of the figure. To compute the edge-strength function by equations (17), we set $\sigma = \rho = 8$ and picked three different values of α, obtaining three different edge-strength functions, v_a, v_b and v_c depicted in the figure. The value of α corresponding to v_a is sufficiently low so that only the corners portions of the central dark area are prominent in the image of v_a. The inner details such as the inner boundaries of the four disjoint corner lobes are smoothed over to a large degree. The value of α was doubled to obtain v_b and doubled again for v_c. At each stage, the edge-strength function becomes more detailed. The figure shows the level curves superimposed on the original image, the set K^+ and its thresholded version as a substitute for the pruned skeleton as explained above. The effect of increasing α is clearly seen in the bottom row of the figure which shows our substitute skeleton. The skeleton in the case of v_a is essentially that of an incomplete rectangle. As α is increased, it becomes more detailed so that the skeleton from v_c depicts the axes of the four lobes more accurately and finds a center for each lobe.

6. References

[1] L. Alvarez, F. Guichard, P.L. Lions and J.M. Morel: ”Axioms and Fundamental Equations of Image Processing”, Arch.for Rat.Mech. (1993).

[2] L. Ambrosio and V.M. Tortorelli: ”On the Approximation of Functionals depending on Jumps by Quadratic, Elliptic Functionals”, Boll. Un. Mat. Ital. (1992)

[3] V. Caselles, F. Catte, T. Coll and F. Dibos: ”A Geometric Model for Active Contours”, Numerische Mathematik 66, (1993).

[4] V. Caselles, R. Kimmel and G. Sapiro: ”Geodesic Active Contours”, Fifth International Conference on Computer Vision, (1995).

[5] B.B. Kimia, A. Tannenbaum and S.W. Zucker: ”On the Evolution of Curves via a Function of Curvature. I. The Classical Case”, J. Math. Anal. and Appl. v.163, no.2, (January, 1992).

[6] B.B. Kimia, A. Tannenbaum and S.W. Zucker: ”Shapes, Shocks and Deformations I: The Components of Shape and the Reaction-Diffusion Space”, International Journal of Computer Vision, (1992).

[7] B.B. Kimia, A. Tannenbaum and S.W. Zucker: ”Entropy Scale-Space”, in *Visual Form*, Ed. C. Arcelli et al, Plenum.

[8] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum and A. Yezzi: ”Gradient Flows and Geometric Active Contour Models”, Fifth International Conference on Computer Vision, (1995).

[9] R. Malladi, J.A. Sethian and B.C. Vemuri: ”Shape Modeling with Front Propagation: A Level Set Approach”, IEEE-PAMI 17, (1995).

[10] D. Mumford and J. Shah: ”Boundary detection by minimizing functionals, I”, Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco (1985)

[11] D. Mumford and J. Shah: ”Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems”, Comm. on Pure and Appl.Math., v.XLII, n.5, pp.577-684 (July, 1989).

[12] S. Osher and J. Sethian: ”Fronts Propagating with Curvature Dependent Speed: Algorithms based on the Hamilton-Jacobi Formulation”, J. Comp. Physics, 79, (1988).

[13] P.J. Olver, G. Sapiro and A. Tannenbaum: ”Differential Invariant Signatures and Flows in Computer Vision, A Symmetry Group Approach”, Geometry Driven Diffusion in Computer Vision, Kluwer, (1994).

[14] J. Shah: ”Uses of Elliptic Approximations in Computer Vision”, to appear in *Variational Methods for Discontinuous Structures*, Ed: R. Serapioni and F. Tomarelli, Birkhäuser.

[15] J. Shah: ”Shape Recovery from Noisy Images by Curve Evolution” IASTED International Conf. on Signal and Image Processing, (Nov. 1995).

[16] J. Shah: ”A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion”, Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco (1996).

[17] K. Siddiqi and B.B. Kimia: ”Parts of Visual Form: Computational Aspects”, IEEE Conf. on Vision and Pattern Recognition, (1993).

[18] K. Siddiqi and B.B. Kimia: ”A Shock Grammar for Recognition”, Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco (1996).

[19] H. Tek and B.B. Kimia: ”Image Segmentation by Reaction-Diffusion Bubbles”, Fifth International Conference on Computer Vision, (1995).