ELASTICA WITH HINGES[1]

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Abstract

During the last decade, curve evolution has been applied to shape recovery, shape analysis, image smoothing and image segmentation. Almost all of these applications are based on curve evolution which minimizes the total length of the curve. The curve moves with velocity proportional to the curvature and hence corners are smoothed out very rapidly. However, many of the approaches to shape analysis require corner-preserving presmoothing of shapes. To preserve corners, it is necessary to consider cost functionals based on curvature rather than on total length. Classically, such functionals have been applied to study bending of thin elastic rods called elastica. In this paper, an implementable formulation based on curvature is developed for smoothing curves while preserving corners.

I. Introduction

Methods based on geometry-driven diffusion for processing images have undergone a great deal of development [1]. In particular, methods based on curve evolution have attracted a lot of attention. These methods have been applied to shape recovery and shape analysis as well as to image smoothing and image segmentation. (See [2] for example and references given there.) Almost all applications of curve evolution are based on minimizing the total length of the curve, sometimes with respect to a specially designed metric, so that points on the curve move with velocity proportional to curvature. Hence, the velocity is greatest at the corners and corners of the shape are rapidly rounded out. However, many approaches to shape analysis require corner-preserving presmoothing of the shape. There is no way to introduce corners in the formulation because the length measure is not sensitive to corners. The only way to introduce corners is to have a cost functional based on curvature. Although such functionals have been proposed from time to time, they all suffer from a lack of practical algorithms. In some cases, algorithms have been devised, but only after imposing some form of severe restriction on the curve. The "snake" formulation of Kass, Witkin, Terzopoulos [3] depends on parametrization and corners are not allowed. In [4], Pauwels, Fiddelaers and Van Gool derive an intrinsic formulation only after imposing the condition that the length of the curve does not change during its evolution. This has the effect of grossly distorting the shape if the initial shape boundary is very noisy. A simple example of this is a square shape whose 3 sides are perfectly straight while the fourth is very wiggly. Since the length is preserved, as the wiggles are straightened out, the fourth side expands, distorting the square into a trapezium. Curvature based functionals have also been used in segmentation functionals to represent segmentation boundaries with corners [5]. However here too, algorithms for implementation are lacking. There is a discrete curve simplification algorithm due to L.J. Latecki and R. Lakaemper [6] which produces a hierarchical description of the vertices of the initial polygonal curve in terms of their saliency. However, it is not applicable for obtaining a piecewise smooth approximation of the curve for use in later applications, for example, determining its skeleton.

In this paper, an implementable formulation is developed for smoothing curves while preserving corners. The starting point is a functional which is the integral of the square of the curvature of the evolving curve. Such a curve when evolving while maintaining its total length is classically called an elastica and its theory goes back to Euler [7]. To permit corners, the functional is augmented by a penalty for corners, allowing smoothing to occur only between corners. In order to facilitate implementation by gradient descent, the discrete corner term is then replaced by a continuous variable called the "corner strength function" which may be thought of as a distributed version of the corners. Next, the curve is embedded as a level curve of a function defined over the plane and the functional is extended to the function by integrating the individual functionals of all the level curves of the function. Finally, the functional may be augmented by including a data fidelity term. The final form of the functional may be viewed as a segmentation functional and may be applied to intensity images. A pair of diffusion equations are derived by gradient descent which are implemented by a numerical method introduced by Osher and Sethian [8].

II. Elastica

The starting point is the functional

$$\int_C \kappa^2 ds \tag{1}$$

where C is an evolving planar curve, κ is the curvature and s is the arc-length. A somewhat more general functional considered in [5] is

$$\int_C \left(a^2 + \kappa^2\right) ds \tag{2}$$

where a is a constant. If the curvature term is omitted from the above functional, it reduces to the usual functional of total length. If the length of curve is held constant during the evolution, then $\int a^2 ds$ is constant and hence, the constant a^2 may be ignored. The first variation of the functional with respect to a small displacement $\delta\eta$ along the normals is easily derived:

$$\delta \int_C (a^2 + \kappa^2) ds = \int_C \left\{ \left(a^2 - \kappa^2\right) \kappa - 2\kappa'' \right\} \delta\eta ds \tag{3}$$

where the superscript prime indicates differentiation with respect to the arc-length s. Here, normals are assumed to be outward and the curvature is defined so that it is positive when C is a circle. The stationary curves satisfy the equation

$$2\kappa'' - \left(a^2 - \kappa^2\right) \kappa = 0 \tag{4}$$

There are infinitely many solutions of this equation which have been described by Mumford [9]. However, it follows from a theorem of Wen [10] that if the stationary curve is a simple, closed, smooth curve, it must be a circle of radius $1/a$. In fact it represents the unique minimum of the functional. The two examples of elastica (stationary curves) in [9] have singular points and thus have infinite energy.

The gradient descent equation for C is

$$\frac{\partial \eta}{\partial t} = 2\kappa'' - \left(a^2 - \kappa^2\right)\kappa \tag{5}$$

The evolution of C is markedly different from that of a curve evolving to minimize its length. Consider the special case in which the initial curve is a perfect circle. Then the motion of the curve is circularly symmetric and the evolution equation reduces to

$$\frac{dr}{dt} = \frac{1}{r}\left(\frac{1}{r^2} - a^2\right) \tag{6}$$

where r is the instantaneous radius. Therefore, if $r < \frac{1}{a}$, the circle expands and if $r > \frac{1}{a}$, it contracts towards its steady state of the circle of radius $1/a$. In particular, if $a = 0$, the curve will expand forever, but at a steadily slower rate. This is a decidedly different behavior from the evolution which minimizes the total length of the curve. In that case, as the curve shrinks, its evolution accelerates and the curve disappears in a finite amount of time. One has to introduce a stopping term to obtain a non-trivial steady state.

When C is not a circle, the κ''–term is negative where the curvature is maximum and thus has the effect of pushing the curve inward. The opposite happens where the curvature is minimum. Thus the effect of the term is to make C more circular. After the small-scale features are smoothed out, the evolution is very slow and the curve may be thought to be practically in a steady state.

The next step is to introduce corners. Following the example of segmentation functionals, the obvious functional to consider is

$$\int_{C\setminus N} \left(a^2 + \kappa^2\right) ds + \nu\,|N| \tag{7}$$

where N is a discrete set of points and $|N|$ denotes the cardinality of N. As mentioned before, it is not possible to apply straight–forward gradient descent to this functional due to the presence of the discrete variable N. Therefore we replace the discrete corners by a continuous "corner strength function" w as follows:

$$\int_C (1-w)^2 \,(a^2 + \kappa^2) ds \tag{8}$$

where function w, defined over the plane, varies between 0 and 1. The higher the curvature, the higher the value of w. At the corners of C, its value equals 1. The strategy for determining w is discussed in the next section. The modified evolution equation now reads

$$\frac{\partial \eta}{\partial t} = 2(\theta\kappa)'' - \theta\kappa\left(a^2 - \kappa^2\right) - \frac{\partial \theta}{\partial n}\left(a^2 + \kappa^2\right) \tag{9}$$

where $\theta = (1-w)^2$ and n indicates the direction normal to the curve.

III. Evolution of the level curves of a surface

It is impractical to implement evolution of C in the form of Eq. (9). Instead, we follow Osher and Sethian [8] and obtain evolution of C as a level curve of an evolving function u. We let all the level curves of u evolve simultaneously in accordance with Eq. (9). The corresponding functional for u is easily obtained by means of the coarea formula. Let u be defined over a domain Ω. Then the total energy of all the level curves combined is

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\int_{u=c} (1-w)^2 \left(a^2+\kappa^2\right) ds \right] dc && (10) \\ = & \int_{\Omega} (1-w)^2 \left(a^2+\kappa^2\right) \|\nabla u\| && (11) \end{aligned}$$

where κ is the curvature of the level curves of u, given by the formula

$$\kappa = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{\|\nabla u\|^3} \tag{12}$$

The coarea formula is also useful for deriving the first variation. With $\theta = (1-w)^2$,

$$\begin{aligned} & \delta \int_{\Omega} \theta \left(a^2+\kappa^2\right) \|\nabla u\| && (13) \\ = & \int_{-\infty}^{\infty} \left[\delta \int_{u=c} \theta \left(a^2+\kappa^2\right) ds \right] dc && (14) \\ = & \int_{-\infty}^{\infty} \left[\int_{u=c} \left\{ -2(\theta\kappa)'' + \theta\kappa \left(a^2-\kappa^2\right) + \frac{\partial \theta}{\partial n} \left(a^2+\kappa^2\right) \right\} \delta\eta ds \right] dc && (15) \\ = & \int_{\Omega} \left\{ 2(\theta\kappa)'' - \theta\kappa \left(a^2-\kappa^2\right) - \frac{\partial \theta}{\partial n} \left(a^2+\kappa^2\right) \right\} \delta u && (16) \\ = & \int_{\Omega} \left\{ 2(\theta\kappa)'' - \theta\kappa \left(a^2-\kappa^2\right) - \nabla\theta \cdot \frac{\nabla u}{\|\nabla u\|} \left(a^2+\kappa^2\right) \right\} \delta u && (17) \end{aligned}$$

where we have used the fact that $\delta\eta||\nabla u|| = -\delta u$.

Functional (11) may be generalized further by adding a penalty for the corner strength function making its computation intrinsic. Morever, a data fidelity term may be added to ensure existence of nontrivial steady states. These terms may be carried over from the segmentation functional based on length-minimizing curve evolution [2]. Let the initial curve be a level curve of a function g. The final form of the functional is

$$E(u,w) = \int_{\Omega} \left[\alpha (1-w)^2 \left(a^2+\kappa^2\right) \|\nabla u\| + \beta |u-g| + \frac{\rho}{2} \|\nabla w\|^2 + \frac{w^2}{2\rho} \right] \tag{18}$$

Preliminary considerations suggest that as $\rho \to 0$, $E(u,w)$ converges to the functional

$$E(u,\Gamma_1,\Gamma_2) = \int_{\Omega\setminus(\Gamma_1\cup\Gamma_2)} \alpha\left(a^2+\kappa^2\right)\|\nabla u\| + \beta\int_\Omega |u-g| + \int_{\Gamma_1} \frac{\alpha\left(a^2+\kappa^2\right)J_u}{1+\alpha\left(a^2+\kappa^2\right)J_u} + |\Gamma_2| \tag{19}$$

where Γ_1 is the locus of discontinuity of u, J_u is the jump in u across Γ_1, Γ_2 is the locus of the corners of the level curves of u and $|\Gamma_2|$ is its length. Note that if the curvature term is omitted from Functional (18), it reduces to the functional given in [2]. It is interesting that like the weak plate model of Blake and Zisserman, functional $E(u,\Gamma_1,\Gamma_2)$ incorporates jumps and creases. However, the Blake-Zisserman model includes all creases, not just the loci of the corners of the level curves. But it is not possible to track evolution of the level curves of u in the Blake and Zisserman model, nor is it possible to implement it by a system of diffusion equations since its elliptic approximation like the Ambrosio-Tortorelli approximation of the Mumford-Shah functional is not known.

With the first variation of the first term in the integrand already given by Eq. (17), the gradient descent equations with respect to u and w are now straightforward to write down:

$$\frac{\partial u}{\partial t} = \left[\left(a^2+\kappa^2\right)\nabla\theta\cdot\frac{\nabla u}{\|\nabla u\|} + \theta\kappa\left(a^2-\kappa^2\right) - 2\left(\theta\kappa\right)'' - \frac{1}{\sigma^3}\frac{u-g}{|u-g|}\right]\|\nabla u\| \tag{20}$$

where $\theta = (1-w)^2$, $\sigma^3 = \alpha/\beta$ and for any function f on Ω, its second derivative with respect to the arc-length along the level curves of u is given by the formula

$$f'' = \frac{u_y^2 f_{xx} - 2u_x u_y f_{xy} + u_x^2 f_{yy}}{\|\nabla u\|^2} - \kappa\frac{\nabla u\cdot\nabla f}{\|\nabla u\|} \tag{21}$$

The gradient descent with respect to w is given by

$$\frac{\partial w}{\partial t} = \nabla^2 w - \frac{w}{\rho^2} + \frac{2\alpha}{\rho}(1-w)\left(a^2+\kappa^2\right)\|\nabla u\| \tag{22}$$

Parameters ρ and σ have the dimension of length. Level curves of u move in accordance with Eq. (9) augmented by a constant velocity component induced by the data fidelity term. As in [2], the corner strength function w is a nonlinear smoothing of

$$\frac{2\alpha\left(a^2+\kappa^2\right)\|\nabla u\|}{1+2\alpha\left(a^2+\kappa^2\right)\|\nabla u\|} \tag{23}$$

Remark: In the derivation above, the penalty for the corner strength function w was introduced after the formulation for surfaces was derived. In order to obtain closer correspondence between Eqs. (7) and (8) one could introduce the penalty for corners in Eq. (8) itself and then derive the formulation for surfaces. Modified Eq. (8) takes the form

$$\int_C \left[\alpha\,(1-w)^2\,(a^2+\kappa^2) + \frac{\rho}{2}\,\|\nabla w\|^2 + \frac{w^2}{2\rho} \right] ds \tag{24}$$

where $\alpha = 1/\nu$. The corresponding functional for surfaces is

$$\int_\Omega \left[\left\{ \alpha\,(1-w)^2\,\left(a^2+\kappa^2\right) + \frac{\rho}{2}\,\|\nabla w\|^2 + \frac{w^2}{2\rho} \right\} \|\nabla u\| + \beta\,|u-g| \right] \tag{25}$$

But now, the gradient descent equations are more complicated. A likely candidate for the limit functional as $\rho \to 0$ is

$$E\,(u, \Gamma_2) = \int_{\Omega\setminus\Gamma_2} \alpha\,\left(a^2+\kappa^2\right) \|\nabla u\| + \beta \int_\Omega |u-g| + \int_{\Gamma_2} \|\nabla u\| \tag{26}$$

IV. Experiments

In the examples below, the constant a was set equal to zero and each time step during diffusion was equal to 0.02.

The first figure illustrates the effect of the corner strength function on the smoothing process. The successive stages of diffusion according to Eq. (20) with $\beta = w = 0$ (thus turning off the fidelity term and the corner-strength function) for all time $t \geq 0$ are shown except in the last frame at the bottom right. Frames from left to right and top to bottom correspond to times t = 0, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. At time t=0, each level curve is a square. As expected, the small inner squares are rounded out rapidly which then expand outward. The outer bigger squares have their corners rounded out, but once the curvature is fairly large, (say for example, radius of curvature > 8 pixels), the diffusion is extremely slow.

The last frame (bottom-right) shows the corner strength function w when the same image is allowed to diffuse according to Eqs. (20) and (22) with the corner term function w included (maximum value of w is equal to about 0.90). Hardly any smoothing takes place. The corner strength function concentrated near the corners prevents diffusion at the corners and there is no diffusion elsewhere because the curvature is zero there.

Examples of smoothing of shape boundaries are shown in the second figure. The corresponding times from left to right are t=0, 60, 600. Each noisy shape boundary was embedded in a surface by means of the signed distance transform. To keep the corner strength function concentrated near the corners, ρ was set equal to 4 pixels. The results were not sensitive to the value of σ which was also set equal to 4 pixels. The really important parameter is α. If it is set too low, values of w will be near zero and the corners will be smoothed out. If the parameter is set too high, w will have values near one everywhere along the boundary because of the high curvature induced by the noise and the diffusion will be very slow. As a compromise, α was set automatically internally during the diffusion so that the maximum value of w was

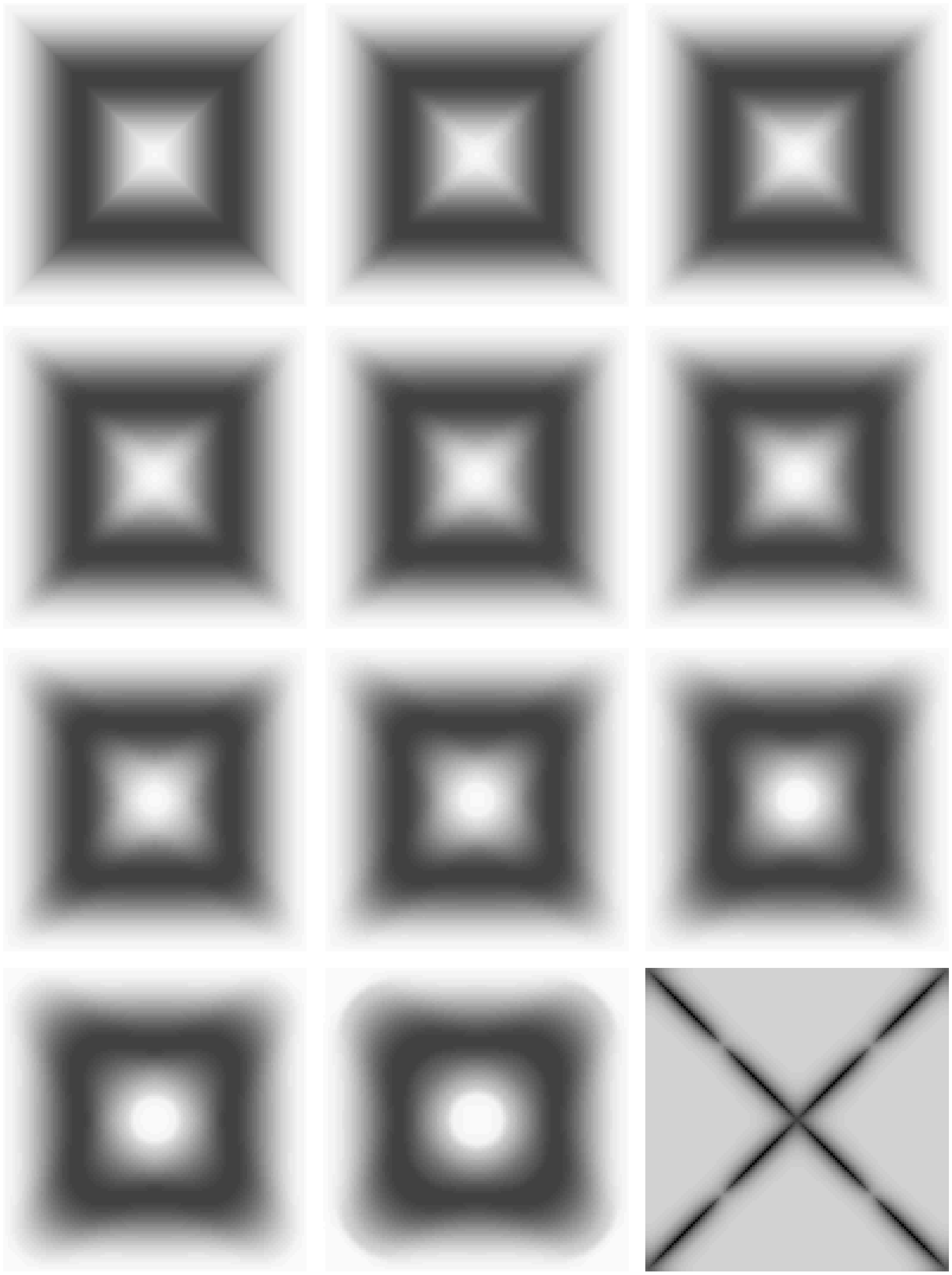

Figure 1: Diffusion according to Eqs. (20) and (22)

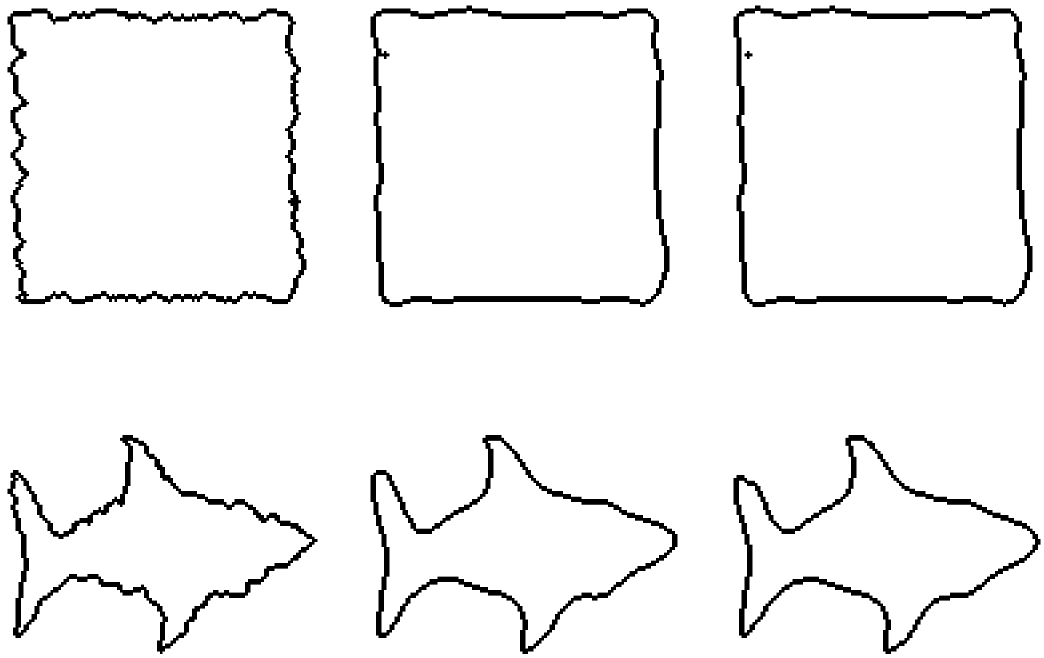

Figure 2: Smoothing of shapes according to Eqs. (20) and (22)

maintained near 0.90. The examples clearly show the effectiveness of the method in smoothing out small-scale noise while preserving corners and large-scale features.

It is not practical to use the method to smooth large-scale features because the diffusion is very slow when the curvature is small. To obtain a smoother shape while preserving corners, a two-step procedure may be followed. Apply Eqs. (20) and (22) as the first step in order to smooth out small–scale noise and save the corner strength function. Once the shape has practically reached a steady state, apply the length-minimizing curve evolution in the presence of the corner function as the second step; that is, let it evolve according to the gradient descent equation

$$\frac{\partial u}{\partial t} = \left[\nabla\theta \cdot \frac{\nabla u}{\|\nabla u\|} + \theta\kappa - \frac{1}{\sigma}\frac{u-g}{|u-g|} \right] \|\nabla u\| \tag{27}$$

keeping θ fixed at values reached during the first stage. The result is a rapid smoothing of the curve away from the corners.

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V. Biography

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