PIECEWISE SMOOTH APPROXIMATIONS OF FUNCTIONS

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Section 1 1. INTRO

1. INTRODUCTION

The motivation for this paper comes from the segmentation problem in Computer Vision which is the problem of subdividing an image into "meaningful" regions ("objects"). During the early stages of image processing, one adopts a very simple criterion for deciding what constitutes a meaningful region: a region is meaningful if it has relatively uniform feature intensity. A mathematical formulation of the problem was proposed in [10] in the form of an "energy functional" as follows:

$$E(u, B) = \frac{1}{\sigma^2} \int_R (u-g)^2 + \int_{R \setminus B} \|\nabla u\|^2 + \lambda |B| \tag{1}$$

where

$R \subset \mathbf{R}^2$ is a connected, bounded, open subset, the "image" domain,
g is the feature intensity, $g : R \to \mathbf{R}$,
$B \subset R$ is a curve segmenting R; it is the union of "object" boundaries,
u is the smoothed image which need not be continuous across B,
$|B|$ is the length of B and
σ, λ are the weights.

The task is to find u and B which minimize $E(u, B)$. The segmentation problem is thus reduced to the problem of finding a piecewise smooth approximation of g. The functional $E(u, B)$ has been studied extensively and a large body of results has been obtained [2,5,6,11,12,13]. The purpose of this paper is to present a modification of this formulation so as to remedy certain "pathologies" exhibited by the above functional. In particular, consider the 1–dimensional version of the problem so that R is just an open interval and B is a finite set of breakpoints. Assume that g is linear. Since the formulation tries to produce an optimal, piecewise smooth and relatively "flat" approximation of g, it will segment R into many small pieces if λ is sufficiently small. One should expect a region in a 2–dimensional image where the image gradient is high to be broken up into small narrow strips. Another problem is that in a 2–dimensional image, the optimal solution always rounds out corners and distorts T-junctions [11]. It is also possible that B may have free ends which curl up into cusps with infinite cuvature at their tips [11]. There is also sensitivity to the shape of a region. For example, consider the situation where the image contains an object D in the shape of a dumbbell with a short narrow neck. Suppose that within D, g is constant in the direction perpendicular to the neck and linear in the other direction. If the neck of D is sufficiently narrow, it will be cost-effective to segment D across the neck. However, if g is constant in the direction of the neck and linear in the other direction, it will never be cost-effective to place a cut across the neck.

In order to avoid these problems, we consider the same functional with boundary conditions:

$$u^{\pm} = g^{\pm} \text{ along } B \tag{2}$$

where the superscripts $+$ and $-$ refer to the values of u and g on the two sides of B. Immediately, we see that B may be assumed to be contained in the discontinuity locus of g. It follows that the singularities such as corners and T-junctions are preserved and that the continuous parts of g are not segmented. Moreover, it is shown in the paper that if the discontinuity locus of g is the union of closed loops (that is, no free ends), then so is the discontinuity locus of u. However, the new functional is quite useless in its present form for application to computer vision. It is quite sensitive to noise since it must agree with g along B. More importantly, if the image is even slightly blurred as it is likely to be in practice, B will be empty since g is now continuous everywhere.

Our solution to these problems is twofold. First we impose the Dirichlet boundary conditions only approximately by constructing a new functional which is in effect an interpolation between the original functional and the one with Dirichlet boundary conditions. Second, we introduce a model for blurred object boundaries in the functional itself. The singular perturbation introduced by Ambrosio and Tortorelli [3,4] provides means to do so. It amounts to replacing B by a continuous variable which may be viewed either as a blurring of B or as the probability for the presence of a boundary at a given point. The method of gradient descent applied to the resulting elliptic functional yields a coupled system of two diffusion equations which may be implemented, for instance, by the method of finite differences. Alternate possibilities for such coupled systems are discussed in [14].

The next section contains the statements of the theorems. The main results are the approximation theorems (in particular, Theorem 7) which are proved only for the 1–dimensional case. The remaining sections are devoted to proving these theorems.

2. STATEMENTS OF THE THEOREMS

We begin with the general n–dimensional framework. Let R be a connected, bounded, open subset of $\mathbf{R}^n$ with Lipschitz boundary. Let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure on $\mathbf{R}^n$, and $\mathcal{H}^{n-1}$ denote the (n-1)-dimensional Hausdorff measure on $\mathbf{R}^n$. Let u and g vary in $L^\infty(R)$, endowed with L^2 topology. Let S_u denote the "jump" set of u as defined in [1]. We make a further regularity assumption on g as follows:

Assumption on g: S_g is contained in a closed subset $K_g \subset R$ such that

(i) $\mathcal{H}^{n-1}(K_g) < \infty$

(ii) $R \backslash K_g$ is a finite union of open sets $\{R_i\}$ with Lipschitz boundary

(iii) $\forall i$, $g|_{R_i}$ is continuous and may be extended continuously to ∂R_i, the boundary of R_i.

Define a class of "piecewise H^1–smooth" functions as

$$PH^1(R) = \left\{ u \in L^\infty(R) \ : \ u|_{R\backslash \bar{S}_u} \in H^1\big(R\backslash \bar{S}_u\big) \quad \text{and } \mathcal{H}^{n-1}\big(\bar{S}_u \cap R\backslash S_u\big) = 0 \right\} \quad (3)$$

where $\bar{S}_u$ denotes the closure of S_u in $\mathbf{R}^n$. Let

$$PH^1(R,g) = \left\{ u \in PH^1(R) : S_u \subset S_g,\ u^\pm = g^\pm \ \ \mathcal{H}^{n-1} - a.e. \text{ along } S_u \right\} \quad (4)$$

We look for the minimizers of the functional

$$E(u) = \int\limits_{R \setminus \bar{S}_u} \left[\frac{1}{\sigma^2}(u-g)^2 + \|\nabla u\|^2 \right] d\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(S_u) \tag{5}$$

in $PH^1(R,g)$.
In order to find approximate minimizers of $E(u)$, we consider the functional

$$\begin{aligned} E_M(u) = & \int\limits_{R \setminus \bar{S}_u} \left[\frac{1}{\sigma^2}(u-g)^2 + \|\nabla u\|^2 \right] d\mathcal{L}^n \\ & + \int\limits_{S_u} \left[\lambda + M\varphi(u^+, u^-, g^+, g^-) \right] d\mathcal{H}^{n-1} \end{aligned} \tag{6}$$

where $u \in PH^1(R)$, M is a constant and

$$\begin{aligned} \varphi(a^+, a^-, b^+, b^-) = \min\{ & (a^+ - b^+)^2 + (a^- - b^+)^2, \\ & (a^+ - b^+)^2 + (a^- - b^-)^2, (a^+ - b^-)^2 + (a^- - b^-)^2 \} \end{aligned} \tag{7}$$

Since we are dealing with piecewise continuous functions, the proper setting for the theory is the space of Special Functions of Bounded Variation or SBV functions, introduced by Ambrosio and may be briefly described as follows [1]. The space $BV(R)$ is the space of functions $u \in L^1(R)$ such that the distributional derivative Du is a vector-valued measure of finite total variation. The space $SBV(R)$ is the space of functions $u \in BV(R)$ such that Du admits the following decomposition: $\forall$ Borel sets V,

$$Du(V) = \int\limits_V \nabla u d\mathcal{L}^{\mathbf{n}} + \int\limits_{V \cap S_u} (u^+ - u^-) \nu_u d\mathcal{H}^{n-1} \tag{8}$$

where ν_u denotes the direction normal to S_u. Let

$$SBV(R,g) = \{ u \in SBV(R) : S_u \subset S_g, \;\; u^{\pm} = g^{\pm} \; \mathcal{H}^{n-1} - a.e. \text{ along } S_u \} \tag{9}$$

Definitions of the functionals $E(u)$ and $E_M(u)$ clearly extend to any $u \in SBV(R)$. We now extend these definitions to any $u \in L^{\infty}(R)$ by relaxation as follows [7]:

$$\bar{E}(u) = \inf \left\{ \liminf_{k \to \infty} E(u_k) : u_k \to u \quad \text{in } L^2(R), u_k \in SBV(R) \right\} \tag{10}$$

$$\bar{E}_{DR}(u) = \inf \left\{ \liminf_{k \to \infty} E(u_k) : u_k \to u \quad \text{in } L^2(R), u_k \in SBV(R,g) \right\} \tag{11}$$

$$\bar{E}_M(u) = \inf \left\{ \liminf_{k \to \infty} E_M(u_k) : u_k \to u \quad \text{in } L^2(R), u_k \in SBV(R) \right\} \tag{12}$$

The following theorem, summarizing the results in [1,8], forms a basis for the results in this paper:

Theorem 1: Assume that $\bar{E}(u) < \infty$. Then, $u \in SBV(R)$ and $\bar{E}(u) = E(u)$. Moreover, $\bar{E}(u)$ has minimizers and any minimizer of $\bar{E}(u)$ belongs to $PH^1(R)$.

Analogously, we prove:

Theorem 2: Assume that $\bar{E}_{DR}(u) < \infty$. Then, $u \in SBV(R, g)$ and $\bar{E}_{DR}(u) = E(u)$.

Theorem 3: Let n=1. If $\bar{E}_M(u) < \infty$, then $u \in PH^1(R)$ and $\bar{E}_M(u) = E_M(u)$.

We have the following existence theorems:

Theorem 4:

(i) $\bar{E}_{DR}(u)$ has a minimizer in $SBV(R, g)$

(ii) If n=1, then $E(u)$ has a minimizer in $PH^1(R, g)$.

(iii) If n=2, suppose that R is a rectangle. Let T_g denote the set consisting of the points in $\bar{S}_g$ which are not in S_g or where $\bar{S}_g$ is not C^1. Similarly, define T_u for any $u \in PH^1(R)$. Suppose that T_g is a finite set. Then $E(u)$ has a minimizer in the set $\{u \in PH^1(R, g) : T_u$ finite$\}$. Moreover, if u is such a minimizer, then $T_u \subset T_g$.

Theorem 5: Let n=1. Then $E_M(u)$ has a minimizer in $PH^1(R)$.

We now construct a singular perturbation of $E_M(u)$. We recall the construction of Ambrosio and Tortorelli. The basic ingredient is the representation of S_u by a continuous variable v as follows. Let

$$\Lambda_\rho = \frac{1}{2} \int\limits_R \left[\rho \|\nabla v\|^2 + \frac{v^2}{\rho} \right] d\mathcal{L}^n \tag{13}$$

For a fixed, (n-1)-dimensional, closed, C^1–smooth subset K in R, let v_ρ minimize $\Lambda_\rho(v)$ with the boundary condition $v = 1$ on K. Then, as $\rho \to 0$, v_ρ obviously tends to zero everywhere in $R \backslash K$. The key point is that $\Lambda_\rho(v_\rho) \to \mathcal{H}^{n-1}(K)$. The values of v_ρ range between 0 and 1. For sufficiently small values of ρ, v_ρ is essentially an exponentially decaying function of distance from K. Thus v_ρ may be viewed as either a blurring of χ_K (the characteristic function of K) with a nominal blurring radius equal to ρ or, alternatively, as the probability for the presence of a boundary at a point. By replacing S_u everywhere in $E_M(u)$ appropriately, we get

$$\begin{aligned} E_{M,\rho} = \int\limits_R [\frac{1}{\sigma^2}(u-g)^2 + \|\nabla u\|^2 \left\{ (1-v)^2 + o_\rho \right\} \\ + \left\{ \frac{1}{2}\lambda + M(u-g)^2 \right\} \left\{ \rho \|\nabla v\|^2 + \frac{v^2}{\rho} \right\}] d\mathcal{L}^n \end{aligned} \tag{14}$$

where o_ρ is a positive constant, depending on ρ, such that o_ρ is of order $o(\rho)$. The infinitesimal o_ρ is introduced as in [4] to ensure C^1–regularity of the minimizers of $E_{M,\rho}$ and thus avoid certain purely technical complications; $E_{M,\rho}$ still approximates E_M if o_ρ

were set equal to 0. The existence of minimizers of $E_{M,\rho}$ in $H^1(R) \times H^1(R)$ follows by reduction to the 1–dimensional case by the slicing technique and then applying the standard compactness and lower semicontinuity theorems (see [3]).

In order to state the approximation theorems, we recall De Giorgi's notion of Γ–convergence [7]. Let X be a metric space, let $Y \subset X$ and let $f_k : Y \to [0, \infty]$ be a family of functions indexed by $k > 0$. Then f_k is said to Γ–converge to $f : X \to [0, \infty]$ as $k \to 0$, if

$$\begin{aligned} &\forall x_k \to x, \liminf_{k \to 0} f_k(x_k) \geq f(x) \\ &\exists x_k \to x, \limsup_{k \to 0} f_k(x_k) \leq f(x) \end{aligned} \tag{15}$$

for all $x \in X$. If the Γ–limit exists, it is unique and it is lower semicontinous. Note that if $\forall k, f_k = f$, then $\Gamma - \lim_{k \to 0} f_k$ exists and equals $\bar{f}$, the relaxation of f. The important property of Γ–convergence from the point of view of approximations is the following: if $\{x_k\}$ is asymptotically minimizing, i.e.,

$$\lim_{k \to 0} \left(f_k(x_k) - \inf_X f_k \right) = 0 \tag{16}$$

and if $x_{k_m} \to x$ for some sequence $k_m \to 0$, then x minimizes f.

The approximation theorems are:

Theorem 6: Let n = 1.

(i) As $M \to \infty$, E_M Γ–converges to $\bar{E}_{DR}$.

(ii) As $M \to 0$, E_M Γ–converges to $\bar{E}$.

Theorem 7: Let n = 1. Let

$$\begin{aligned} X(R) &= L^\infty(R) \times \{v \in L^\infty(R) \ : \ 0 \leq v \leq 1\} \\ Y(R) &= \big\{(u, v) \in X(R) \ : \ u, v \in H^1(R)\big\} \end{aligned} \tag{17}$$

Extend $\bar{E}_M$ to $X(R)$ by setting

$$\bar{E}_M(u, v) = \begin{cases} \bar{E}_M & \text{if } v \equiv 0 \\ \infty & \text{otherwise} \end{cases} \tag{18}$$

Then as $\rho \to 0$, $E_{M,\rho}$ Γ–converges to $\bar{E}_M$.

Theorem 8: Let n = 1. Let $X(R)$ and $Y(R)$ be defined and $\bar{E}_{DR}$ extended to $X(R)$ as in Theorem 7. Assume that $M = O|\ln \rho|$ and $g \in C^{0,\alpha}(R \backslash \bar{S}_g)$ for some positive $\alpha < \frac{1}{2}$. Then as $\rho \to 0$, $E_{M,\rho}$ Γ–converges to $\bar{E}_{DR}$.

3. PROOF OF THEOREM 2

By Theorem 1, $u \in SBV(R)$ and $E(u) < \infty$. We have to show that u satisfies the boundary conditions. Let $\{u_k\}$ be a sequence in $SBV(R, g)$ such that $u_k \to u$

in L^2. Since $S_{u_k} \subset S_g$ and $(u_k^+ - u_k^-)\nu_{u_k} \cdot \mathcal{H}^{n-1} \llcorner S_{u_k}$ converges in measure [1] to $(u^+ - u^-)\nu_u \cdot \mathcal{H}^{n-1} \llcorner S_u$, $S_u \subset S_g$. Let K_g be a closed subset containing S_g such that $R \backslash K_g$ has finitely many connected components, $\{R_i\}$, with Lipschitz boundary. Since the embedding $H^1(R_i) \subset W_2^s(R_i)$ is compact [9] for $s < 1$, after replacing the sequence u_k by a subsequence, we may assume that $u_k \to u$ in $W_2^s(R_i), \forall i$. If $s - \frac{1}{2} > 0$, then we have continuous trace maps $\gamma_i : W_2^s(R_i) \to W_2^{s-1/2}(\partial R_i)$. Therefore, $\gamma_i(u_k) \to \gamma_i(u)$ in $L^2(\partial R_i), \forall i$. It follows that along K_g, the function

$$\begin{aligned} 0 = &\left(\left|u_k^+ - g^+\right| + \left|u_k^- - g^-\right|\right)\left(\left|u_k^+ - u_k^-\right|\right) \to \\ &\left(\left|u^+ - g^+\right| + \left|u^- - g^-\right|\right)\left(\left|u^+ - u^-\right|\right) \quad \mathcal{H}^{n-1} - a.e. \end{aligned} \tag{19}$$

Hence, $u^\pm = g^\pm \quad \mathcal{H}^{n-1} - a.e.$ along S_u.

4. PROOF OF THEOREM 3

Again, by Theorem 1, $u \in H^1(R/\bar{S}_u)$ and $\bar{S}_u \backslash S_u$ is empty since n=1. The main point is to show that $\bar{E}_M(u) \geq E_M(u)$. Let $\{u_k\}$ be a sequence in $PH^1(R)$ converging to u in $L^2(R)$. By passing to a subsequence, we may assume that $\nabla u_k \to \nabla u$ weakly in $L^2(R)$, $|S_{u_k}| = N, \forall k$ and

$$\lim_{k \to 0} E_M(u_k) = \bar{E}_M(u) \tag{20}$$

By Theorem 1, $\liminf\limits_{k \to 0} E(u_k) \geq E(u)$. We have to show that

$$\liminf_{k \to 0} \sum_{S_{u_k}} \varphi(u_k^+, u_k^-, g^+, g^-) \geq \sum_{S_u} \varphi(u^+, u^-, g^+, g^-) \tag{21}$$

By Ambrosio's semicontinuity theorem [1], for every open set $A \subset R$, $|S_u \cap A| \leq \liminf\limits_{k \to 0} |S_{u_k} \cap A|$. Let R be the open interval (a, b) and let

$$\begin{aligned} &S_{u_k} = \{x_{k_i}\}, \;\; a < x_{k_1} < x_{k_2} \cdots < x_{k_N} < b \\ &S_u = \{y_j\}, \;\; a < y_1 < y_2 < \ldots < b \end{aligned} \tag{22}$$

Let $\{x_{k_l}, \ldots, x_{k_m}\}$ be the set of points in S_{u_k} converging to a point $y_j \in S_u$. After passing to a subsequence, we may assume that the convergence is monotonic. Define

$$\begin{aligned} v_k(x) = \int_a^x \nabla u_k dx, \quad v(x) = \int_a^x \nabla u dx \\ w_k = u_k - v_k, \quad w = u - v \end{aligned} \tag{23}$$

Then $v_k \to v$ in $C^{0,\alpha}(R)$ for all $\alpha < \frac{1}{2}$, w_k and w are piecewise constant, $w_k \to w \;\; a.e.$ and $Dw_k \to Dw$ weakly in measure [1].

It follows that if we let

$$u_k(x_{k_i}+) \to a_i^+,\ u_k(x_{k_i}-) \to a_i^- \quad \text{for } l \le i \le m \tag{24}$$

then

$$u(y_j+) = a_m^+,\ u(y_j-) = a_l^-,\ a_i^+ = a_{i+1}^- \quad \text{for } l \le i < m \tag{25}$$

Let $\varphi_{k_i} = \varphi(u_k(x_{k_i}+), u_k(x_{k_i}-), g(x_{k_i}+), g(x_{k_i}-))$. We have to show that

$$\sum_{i=l}^{i=m} \varphi_{k_i} \ge \varphi(u(y_j+), u(y_j-), g(y_j+), g(y_j-)) \tag{26}$$

Suppose that $x_{k_l}, x_{k_m} \notin S_g, \forall k \ge$ some k_0. Now if $x_{k_i} \notin S_g, \forall k \ge k_0$, then $u_k(x_{k_i}-) - g(x_{k_i}) \to u(y_j\pm) - g(y_j\pm)$ where the sign on the right side is plus if $x_{k_i} > y_j$ and minus if $x_{k_i} < y_j$. Then it is easily checked that

$$\begin{aligned} &\varphi(u(y_j+), u(y_j-), g(y_j+), g(y_j-)) \\ &\le \begin{cases} \liminf\limits_{k\to 0} \varphi_{k_l} & \text{if } \ l = m \\ \liminf\limits_{k\to 0} (\varphi_{k_l} + \varphi_{k_m}) & \text{otherwise} \end{cases} \end{aligned} \tag{27}$$

Now suppose that there exists an integer k_0 such that $\forall k > k_0, x_{k_l}$ or $x_{k_m} \in S_g$. If $l = m$, then we are done. If $l \ne m$, then, say, $x_{k_l} \in S_g$. Hence, $x_{k_m} \notin S_g$. One checks that

$$\begin{aligned} &\liminf_{k\to 0} (\varphi_{k_l} + \varphi_{k_m}) \\ &\ge \min\left\{ \left[a_l^- - g(y_j-)\right]^2 + \left[a_m^+ - g(y_j+)\right]^2,\ \ \left[a_l^- - g(y_j+)\right]^2 + \left[a_m^+ - g(y_j+)\right]^2 \right\} \\ &\ge \varphi(u(y_j+), u(y_j-), g(y_j+), g(y_j-)) \end{aligned} \tag{28}$$

5. PROOFS OF THEOREMS 4 AND 5

For proving the first part of Theorem 4, let u_k be a minimizing sequence in $SBV(R,g)$ converging to u in $L^2(R)$. Let $c = \|g\|_{\infty,R}$ and let $\bar{u}_k = c \wedge u_k \vee (-c)$. Then $\bar{u}_k \in SBV(R,g)$ and $E(\bar{u}_k) \le E(u_k)$. Hence we may assume that $\|u_k\|_{\infty,R} \le c$. By Theorem 2, $u \in SBV(R,g)$ and $E(u) \le \lim_{k\to 0} E(u_k)$. Therefore u minimizes the functional. If n=1, then u is automatically in $PH^1(R,g)$. Theorem 5 is proved in the same way. We now give a direct proof for part (iii) of Theorem 4. Let $u \in PH^1(R,g)$. Suppose that there exists $P \in T_u$ which is not in T_g. Then, since S_g must be C^1 at P, P must be a terminal point of S_u. By Theorem 1.7.3 in [9], there must exist $\epsilon > 0$ such that with origin at $P, \forall \delta < \epsilon$,

$$\int_0^\delta \left|u^+(s) - u^-(s)\right| \frac{ds}{s} < \infty \tag{29}$$

along the branch of S_u terminating at P. Here s is the arc-length along the branch. Therefore,

$$\int_0^{\delta} \left| g^+(s) - g^-(s) \right| \frac{ds}{s} < \infty \tag{30}$$

But since S_g is C^1 at P, there exists a positive $\delta < \epsilon$ such that

$$\left| g^+(s) - g^-(s) \right| > a > 0 \quad \text{for } 0 < s < \delta \tag{31}$$

and hence

$$\int_0^{\delta} \left| g^+(s) - g^-(s) \right| \frac{ds}{s} = \infty \tag{32}$$

Therefore, P must be in T_g as well. Hence, we need to consider only those functions $u \in PH^1(R, g)$ for which $T_u \subset T_g$. Since T_g is finite, there are only finitely many possibilities for S_u. Hence the theorem follows from the theory of elliptic boundary value problems [9].

6. PROOF OF THEOREM 6

(i) **CASE:** $M \to \infty$

Proposition 6.1 (lower inequality): For every $u \in SBV(R) \cap L^\infty(R)$,

$$\bar{E}_{DR}(u) \leq \inf \left\{ \liminf_{M \to \infty} E_M(u_M) : u_M \to u \quad \text{in } L^2(R),\ u_M \in PH^1(R) \right\} \tag{33}$$

Proof: We may assume that the right hand side of the inequality is finite. Let $\{u_M\}$ be a sequence in $PH^1(R)$ converging to u in $L^2(R)$ such that $E_M(u_M) \leq C < \infty$ and $|S_{u_M}| = N$. By Theorem 1, $u \in PH^1(R)$, $|S_u| \leq N$ and $E(u) \leq C$. It remains to show that $u \in PH^1(R, g)$. As in the proof of Theorem 3, let $\{x_{M_l}, ..., x_{M_m}\} \subset S_{u_M}$ be the set of points, monotonically converging to $y \in S_u$. Suppose that $x_{M_k} \neq y$ for all sufficiently large M. We may assume that g is continuous at x_{M_k} for all M. Then,

$$(u_M(x_{M_k}+) - g(x_{M_k}))^2 + (u_M(x_{M_k}-) - g(x_{M_k}))^2 \leq \frac{C}{M} \to 0 \tag{34}$$

Hence,

$$\lim_{M \to \infty} u_M(x_{M_k}+) = \lim_{M \to \infty} u_M(x_{M_k}-) = \begin{cases} g(y-) & \text{if } x_{M_k} < y \\ g(y+) & \text{if } x_{M_k} > y \end{cases} \tag{35}$$

Therefore, $u_M(x_{M_k}+) - u_M(x_{M_k}-) \to 0$ as $M \to \infty$. Since, by Ambrosio's theorem [1],

$$\sum_k (u_M(x_{M_k}+) - u_M(x_{M_k}-)) \to u(y+) - u(y-) \tag{36}$$

there must exist k such that for all sufficiently large M, $x_{M_k} = y$ and such that $u_M(y+) \to u(y+)$ and $u_M(y-) \to u(y-)$. Since

$$\varphi(u_M(y+), u_M(y-), g(y+), g(y-)) \leq \frac{C}{M} \to 0 \tag{37}$$

it is easy to check that $y \in S_g$ and $u^{\pm}(y) = g^{\pm}(y)$.

Proposition 6.2 (upper inequality): For every $u \in SBV(R)$, there exists a sequence $\{u_M\}$ in $PH^1(R)$ converging to u such that

$$\limsup_{M \to \infty} E_M(u_M) \leq \bar{E}_{DR}(u) \tag{38}$$

Proof: We may assume that $\bar{E}_{DR}(u) < \infty$. Hence, $u \in PH^1(R, g)$. Set $u_M = u, \forall M$.

(ii) **CASE:** $M \to 0$

The lower inequality follow from Ambrosio's theorem [1]. For the upper inequality, set $u_M = u, \forall M$ as before.

7. PROOF OF THEOREM 7

Proposition 7.1 (lower inequality): Let n=1. Then, for every $(u, v) \in X(R)$,

$$\bar{E}_M(u,v) \leq \inf\left\{\liminf_{\rho \to 0} E_{M,\rho}(u_\rho, v_\rho) \ : \ (u_\rho, v_\rho) \to (u,v) \text{ in } L^2(R), \ (u_\rho, v_\rho) \in Y(R)\right\} \tag{39}$$

Proof: If $v \not\equiv 0$, then both sides of the inequality would be infinite. Therefore, we may assume that the $v \equiv 0$ and the right hand side of the inequality is finite. Let $(u_\rho, v_\rho) \to (u, 0)$ in $L^2(R)$ such that $E_{M,\rho}(u_\rho, v_\rho) \leq C < \infty$. By Theorem 1, $u \in SBV(R)$ and $E(u) \leq \liminf_{\rho \to 0} E_{0,\rho}(u_\rho, v_\rho) < \infty$. Hence, $u \in PH^1(R)$. Therefore, it is enough to show that for every positive $\delta < 1$, there exists $\eta > 0$ such that $\{(y - \eta, y + \eta) : y \in S_u\}$ are disjoint open sets in R, and $\forall y \in S_u$,

$$\liminf_{\rho \to 0} \int_{y-\eta}^{y+\eta} (u_\rho - g)^2 \left(\rho |v_\rho'|^2 + \frac{v_\rho^2}{\rho}\right) \geq (1 - \delta)\varphi\big(u^+(y), u^-(y), g^+(y), g^-(y)\big) \tag{40}$$

Ambrosio and Tortorelli [3,4] prove that since

$$\frac{C}{\lambda} \geq \frac{1}{2}\int_R \left\{\rho |v_\rho'|^2 + \frac{v_\rho^2}{\rho}\right\} dx \geq \int_R |v_\rho v_\rho'| dx \geq \frac{1}{2}\int_R \left|(v_\rho^2)'\right| dx \tag{41}$$

there exists a finite set $J \subset R$ such that (i) $S_u \subset J$ and (ii) for any compact set $K \subset R \backslash J$,

$$\limsup_{\rho \to 0} \max_K v_\rho^2 \leq \delta \tag{42}$$

Therefore, after passing to a subsequence if necessary, we can find $\eta > 0$ such that $\{(y-\eta, y+\eta) : y \in S_u\}$ are disjoint open sets in R and

(i) $v_\rho^2(x) < \delta \ \forall \rho$ if $x \notin (y-\eta, y+\eta) \ \forall y \in S_u$

(ii) $\forall y \in S_u$, $(y-\eta, y+\eta) \cap S_g = \begin{cases} \{y\} & \text{if } y \in S_g \\ \emptyset & \text{otherwise} \end{cases}$

(iii) $\forall y \in S_u$, $(y-\eta, y+\eta) \cap J = \{y\}$

For any $\sigma \geq \delta$, let

$$\begin{aligned} y^-_{\rho,\sigma} &= \max_{x \leq y+\eta} \left\{ \sup_{(y-\eta,x)} v_\rho^2 < \sigma \right\} \\ y^+_{\rho,\sigma} &= \min_{x \geq y-\eta} \left\{ \sup_{(x,y+\eta)} v_\rho^2 < \sigma \right\} \end{aligned} \tag{43}$$

Now,

$$\lim_{\rho \to 0} \sup_{(y-\eta, y+\eta)} v_\rho^2 = 1 \tag{44}$$

because otherwise u would be absolutely continuous at y. Hence, $y^-_{\rho,\sigma} < y^+_{\rho,\sigma}$ and the sequences $\{y^-_{\rho,\sigma}\}$ and $\{y^+_{\rho,\sigma}\}$ have a unique accumulation point at y. Fix σ such that $\delta < \sigma < 1$. After extracting a subsequence if necessary, we may assume that the sequences $\{y^-_{\rho,\sigma}\}$, $\{y^+_{\rho,\sigma}\}$, $\{y^-_{\rho,\delta}\}$, $\{y^+_{\rho,\delta}\}$ converge to y monotonically. Define $\underline{u}_{\rho,\sigma}$ over $(y-\eta, y+\eta)$ as follows:

$$\underline{u}_{\rho,\sigma}(x) = \begin{cases} u_\rho(x) & \text{if } x \in \left(y-\eta, y^-_{\rho,\sigma}\right) \\ u_\rho\left(y^-_{\rho,\sigma}\right) & \text{otherwise} \end{cases} \tag{45}$$

Since

$$\int_{y-\eta}^{y+\eta} \left(\underline{u}'_{\rho,\sigma}\right)^2 dx = \int_{y-\eta}^{y^-_{\rho,\sigma}} \left(u'_\rho\right)^2 dx \leq \frac{1}{(1-\sqrt{\sigma})^2} \int_{y-\eta}^{y^-_{\rho,\sigma}} \left(u'_\rho\right)^2 (1-v_\rho)^2 dx \leq \frac{C}{(1-\sqrt{\sigma})^2} \tag{46}$$

as $\rho \to 0$, $\underline{u}_{\rho,\sigma} \to \underline{u}$ in $C^{0,\alpha}(y-\eta, y+\eta)$ for any positive $\alpha < \frac{1}{2}$ where

$$\underline{u}(x) = \begin{cases} u(x) & \text{if } x \in (y-\eta, y) \\ u^-(y) & \text{if } x \in (y, y+\eta) \end{cases} \tag{47}$$

Let $A = \max\left\{\|u\|_{\infty,R}, \|g\|_{\infty,R}\right\}$. Let $\epsilon > 0$ such that $\epsilon < 11A^2$. There exists a positive ρ_0 such that for all $\rho < \rho_0$,

$$\left\|\underline{u}_{\rho,\sigma} - \underline{u}\right\|_{\infty,(y-\eta,y+\eta)} < \frac{\epsilon}{11A} \tag{48}$$

and

$$\left\|\underline{u}_{\rho,\sigma}+\underline{u}\right\|_{\infty,(y-\eta,y+\eta)} \le \left\|\underline{u}_{\rho,\sigma}-\underline{u}\right\|_{\infty,(y-\eta,y+\eta)} + 2\|\underline{u}\|_{\infty,(y-\eta,y+\eta)} \le 3A \tag{49}$$

Let

$$b^- = \begin{cases} u^-(y) - g^-(y) & \text{if } y^-_{\rho,\sigma} < y \\ u^-(y) - g^+(y) & \text{if } y^-_{\rho,\sigma} > y \end{cases} \tag{50}$$

By continuity of $\underline{u}$ and g in $(y-\eta, y+\eta)\backslash\{y\}$, we may choose ρ_0 such that $\forall \rho < \rho_0$,

$$\left|\underline{u}(x) - g(x) - b^-\right| < \frac{\epsilon}{11A} \quad \forall x \in \left(y^-_{\rho,\delta}, y^-_{\rho,\sigma}\right) \tag{51}$$

Then $\forall x \in (y^-_{\rho,\delta}, y^-_{\rho,\sigma})$ and $\forall \rho < \rho_0$,

$$\begin{aligned} &\left|(u_\rho - g)^2 - \left(b^-\right)^2\right| \\ &\le \left|\left(\underline{u}_{\rho,\sigma}-\underline{u}\right)^2 + 2\left(\underline{u}_{\rho,\sigma}-\underline{u}\right)(\underline{u}-g) + (\underline{u}-g)^2 - \left(b^-\right)^2\right| \\ &\le 7A\left|\left(\underline{u}_{\rho,\sigma}-\underline{u}\right)\right| + 4A\left|\underline{u}-g-b^-\right| \\ &\le \epsilon \end{aligned} \tag{52}$$

Therefore, $\forall \rho < \rho_0$,

$$\begin{aligned} &\int_{y-\eta}^{y^-_{\rho,\sigma}} (u_\rho - g)^2 \left\{\rho\left(v'_\rho\right)^2 + \frac{v_\rho^2}{\rho}\right\} dx \\ &\ge \int_{y^-_{\rho,\delta}}^{y^-_{\rho,\sigma}} (u_\rho - g)^2 \left\{\rho\left(v'_\rho\right)^2 + \frac{v_\rho^2}{\rho}\right\} dx \\ &\ge \left\{\left(b^-\right)^2 - \epsilon\right\} \int_{y^-_{\rho,\delta}}^{y^-_{\rho,\sigma}} \left\{\rho\left(v'_\rho\right)^2 + \frac{v_\rho^2}{\rho}\right\} dx \\ &\ge \left\{\left(b^-\right)^2 - \epsilon\right\} \int_{y^-_{\rho,\delta}}^{y^-_{\rho,\sigma}} \left(v_\rho^2\right)' dx \\ &\ge \left\{\left(b^-\right)^2 - \epsilon\right\}(\sigma - \delta) \end{aligned} \tag{53}$$

Similarly, defining b^+ by replacing u^- by u^+ and $y^-_{\rho,\sigma}$ by $y^+_{\rho,\sigma}$ in the definition of b^-, we get

$$\int_{y^+_{\rho,\sigma}}^{y+\eta} (u_\rho - g)^2 \left\{\rho\left(v'_\rho\right)^2 + \frac{v_\rho^2}{\rho}\right\} dx \ge \left\{\left(b^+\right)^2 - \epsilon\right\}(\sigma - \delta) \tag{54}$$

Since ϵ is arbitrary,

$$\int_{y-\eta}^{y+\eta} (u_\rho - g)^2 \left\{ \rho\big(v'_\rho\big)^2 + \frac{v_\rho^2}{\rho} \right\} dx \geq \left\{ \big(b^+\big)^2 + \big(b^-\big)^2 \right\} (\sigma - \delta) \tag{55}$$

Now let $\sigma \nearrow 1$.

Proposition 7.2 (upper semicontinuity): Let n=1. Then, for every $(u, v) \in X(R)$, there exists a sequence $\{(u_\rho, v_\rho)\}$ in $Y(R)$ converging to (u, v) in $L^2(R)$ such that

$$\limsup_{\rho \to 0} E_{M,\rho}(u_\rho, v_\rho) \leq \bar{E}_M(u, v) \tag{56}$$

Proof: We may assume that $\bar{E}_M(u, v) < \infty$. Hence, $v \equiv 0, u \in PH^1(R)$ and $\bar{E}_M(u, v) = E_M(u)$. We adopt the construction of Ambrosio and Tortorelli [4]. For each ρ, fix

$$\begin{aligned} \eta_\rho &= o(\sqrt{\rho}) \\ b_\rho &= O(\rho o_\rho) \\ a_\rho &= -\rho \ln \eta_\rho \end{aligned} \tag{57}$$

Let

$$\omega_\rho(t) = e^{-(t - b_\rho)/\rho} \quad \text{for } b_\rho \leq t \leq b_\rho + a_\rho \tag{58}$$

Note that $\omega_\rho(b_\rho + a_\rho) = \eta_\rho$. We may assume that for each $y \in S_u$,

$$\begin{aligned} (y - 2b_\rho - 2a_\rho, y + 2b_\rho + 2a_\rho) \cap S_u &= \{y\} \\ (y - 2b_\rho - 2a_\rho, y + 2b_\rho + 2a_\rho) \cap S_g &= \{y\} \text{ or } \emptyset \end{aligned} \tag{59}$$

For $y \in S_u$, let $\varphi_y = \varphi\big(u^+(y), u^-(y), g^+(y), g^-(y)\big)$. For each $y \in S_u$, choose a sequence of points $\{y_\rho\}$ as follows:

$$y_\rho = \begin{cases} y & \text{if } \varphi_y = \big(u^+(y) - g^+(y)\big)^2 + \big(u^-(y) - g^-(y)\big)^2 \\ y - (b_\rho + a_\rho) & \text{if } \varphi_y = \big(u^+(y) - g^-(y)\big)^2 + \big(u^-(y) - g^-(y)\big)^2 \\ y + (b_\rho + a_\rho) & \text{if } \varphi_y = \big(u^+(y) - g^+(y)\big)^2 + \big(u^-(y) - g^+(y)\big)^2 \end{cases} \tag{60}$$

Let

$$v_\rho(x) = \begin{cases} 1 & \text{if } |x - y_\rho| \leq b_\rho \text{ for some } y_\rho \\ \omega_\rho(|x - y_\rho|) & \text{if } b_\rho \leq |x - y_\rho| \leq b_\rho + a_\rho \text{ for some } y_\rho \\ \eta_\rho & \text{otherwise} \end{cases} \tag{61}$$

Finally, define continuous functions u_ρ as follows:

If $|x - y_\rho| \leq b_\rho + a_\rho$ for some y_ρ, let

$$u_\rho(x) = \begin{cases} u^-(y_\rho - b_\rho - a_\rho) & \text{if } y_\rho - b_\rho - a_\rho \leq x \leq y_\rho - b_\rho \quad \text{for some } y_\rho \\ u^+(y_\rho + b_\rho + a_\rho) & \text{if } y_\rho + b_\rho \leq x \leq y_\rho + b_\rho + a_\rho \quad \text{for some } y_\rho \\ \text{linear} & \text{in } (y_\rho - b_\rho, y_\rho + b_\rho) \quad \forall y_\rho \end{cases} \tag{62}$$

If $|x - y_\rho| > b_\rho + a_\rho$ for every y_ρ, let $u_\rho(x) = u(x)$

It is now straightforward to check that (see [3,4])

$$\limsup_{\rho\to 0} \int_R \left[(u'_\rho)^2 \Big\{ (1-v_\rho)^2 + o_\rho \Big\} + \frac{1}{\sigma^2}(u_\rho - g)^2 + \frac{\lambda}{2} \Big\{ \rho (v'_\rho)^2 + \frac{v_\rho^2}{\rho} \Big\} \right] dx \le E(u) \tag{63}$$

In particular,

$$\limsup_{\rho\to 0} \int_{y_\rho - b_\rho - a_\rho}^{y_\rho - b_\rho} \Big\{ \rho (v'_\rho)^2 + \frac{v_\rho^2}{\rho} \Big\} dx = \limsup_{\rho\to 0} \int_{y_\rho + b_\rho}^{y_\rho + b_\rho + a_\rho} \Big\{ \rho (v'_\rho)^2 + \frac{v_\rho^2}{\rho} \Big\} dx = 1 \tag{64}$$

By construction, for any $\epsilon > 0$, there exists ρ_0 such that $\forall \rho < \rho_0$,

$$\begin{aligned} &2[u_\rho(x) - g(x)]^2 o_\rho \le \epsilon \quad \text{if } 0 \le |x - y_\rho| \le b_\rho \\ &[u_\rho(x) - g(x)]^2 \le \varphi_y + \epsilon, \quad \text{if } b_\rho \le |x - y_\rho| \le b_\rho + a_\rho \end{aligned} \tag{65}$$

Hence,

$$\limsup_{\rho\to 0} \int_{y_\rho - b_\rho - a_\rho}^{y_\rho + b_\rho + a_\rho} (u_\rho - g)^2 \Big\{ \Big(\rho (v'_\rho)^2 + \frac{v_\rho^2}{\rho} \Big) \Big\} dx \le 2\varphi_y \tag{66}$$

Since $(u_\rho - g)^2 \eta_\rho^2 = o(\rho)$, the proposition follows.

8. PROOF OF THEOREM 8

Lower Inequality: As in the proof of Proposition 7.1, let $(u_\rho, v_\rho) \to (u, 0)$ in L^2 such that $E_{M,\rho}(u_\rho, v_\rho) \le C < \infty$. By Proposition 7.1, for each fixed M,

$$E_M(u, v) \le \liminf_{\rho\to 0} E_{M,\rho}(u_\rho, v_\rho) \le C \tag{67}$$

By letting $M \to \infty$, we see that $\varphi = 0$ for every point in S_u. Hence, $S_u \subset S_g$ and $u \in PH^1(R, g)$.

Upper Inequality: Use the construction given in the proof of Proposition 7.2, noting that $Mo_\rho \to 0$ and $M[u_\rho(x) - g(x)]^2 = O\big(M(b_\rho + a_\rho)^\alpha\big) = O\big(\rho^\alpha \ln^{\alpha+1} \rho\big)$.

REFERENCES

[1] L. Ambrosio: *A compactness theorem for a special class of functions of bounded variation.* Boll. Un. Mat. Ital. 3-B,7, 857-881, 1989.

[2] L. Ambrosio, *Existence theory for a new class of variational problems.* Arch. Rat. Mech. Anal. 111, 291-322, 1990.

[3] L. Ambrosio and V.M. Tortorelli: *Approximation of functionals depending on jumps by elliptic functionals via* Γ*-convergence.* Comm. Pure Appl. Math. 43, 999-1036, 1990.

[4] L. Ambrosio and V.M. Tortorelli: *On the approximation of functionals depending on jumps by quadratic, elliptic functionals.* Boll. Un. Mat. Ital. (to appear)

[5] G. Congedo and I. Tamanini: *On the existence of solutions to a problem in multidimensional segmentation.* Ann. Inst. H. Poincaré, Anal. Nonlin. 8, 175-195, 1991.

[6] G. Dal Maso, J.-M. Morel and S. Solimini: *A variational method in image segmentation: Existence and approximation results.* Acta Math. 168, 89-151, 1992.

[7] G. Dal Maso: *An introduction to* Γ *convergence.* S.I.S.S.A., Trieste, 1992.

[8] E. De Giorgi, M. Carriero and A. Leaci: *Existence theorem for a minimum problem with free discontinuity set.* Arch. Rat. Mech. Anal. 108, 195-218, 1989.

[9] P. Grisvard: *Elliptic problems in nonsmooth domains.* Pitman, 1985.

[10] D. Mumford and J. Shah: *Boundary detection by minimizing functionals, I.* Proc. of the IEEE Conf. on Computer Vision and Pattern Recognition, 22-26, 1985.

[11] D. Mumford and J. Shah: *Optimal approximations by piecewise smooth functions and associated variational problems.* Comm. Pure Appl. Math. 42, 577-685, 1989.

[12] T. Richardson: *Scale independent, piecewise smooth segmentation of images via variational methods.* Ph.D. thesis, Lab. for Info. and Decision Systems, LIDS-Th-1940, MIT, 1990.

[13] J. Shah: *Properties of energy-minimizing segmentations.* SIAM J. Control and Optim. v.30, no.1, 99-111, 1992.

[14] J. Shah: *Segmentation by nonlinear diffusion, II.* Proc. of the IEEE Conf. on Computer Vision and Pattern Recognition, 644-647, 1992.