Approximation of non-convex functionals in GBV

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1 Introduction

In many problems of Computer Vision, the unknown is a pair (u, K) with K varying in a class of (sufficiently smooth) closed hypersurfaces contained in a fixed open set $\Omega \subset \mathbf{R}^2$ and $u : \Omega \setminus K \to \mathbf{R}$ belonging to a class of (sufficiently smooth) functions. A variational formulation of some of these problems was given by Mumford and Shah [12] introducing the functional

$$F(u,K) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + c_1\mathcal{H}^1(K) + c_2\int_{\Omega\setminus K} |u-g|^2\,dx\,. \tag{1}$$

In this case g is interpreted as the input picture taken from a camera, u is the 'cleaned' image, and K is the relevant contour of the objects in the picture; c_1 and c_2 are contrast parameters, and $\mathcal{H}^1(K)$ denotes the total length of K (which is a union of curves). Problems involving functionals of this form are usually called free-discontinuity problems (see [9], [2], [5]).

The presence of the unknown surface K leads to numerical problems, to solve which some kind of approximation of this functional is needed to obtain approximate smooth solutions. The Ambrosio and Tortorelli approach [3] provides a variational approximation of the Mumford and Shah functional (1) via elliptic functionals. The lack of convexity of the limiting functional is overcome by the introduction of an additional function variable which approaches the characteristic of the complement of the jump set. The approximating functionals have the form

$$F_\varepsilon(u,v) = \int_\Omega v^2|\nabla u|^2\,dx + c_1\int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v)^2\Big)\,dx + c_2\int_\Omega |u-g|^2\,dx\,, \tag{2}$$

defined on functions u, v such that $u, v \in H^1(\Omega)$ and $0 \le v \le 1$. The interaction of the terms in the second integral provide an approximate interfacial energy. The adaptation of the Ambrosio and Tortorelli approximation to obtain as limits more complex surface energies does not seem to follow easily from their approach.

In this paper we study a variant of the Ambrosio Tortorelli construction by considering functionals of the form

$$G_\varepsilon(u,v) = \int_\Omega v^2|\nabla u|\,dx + \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v)^2\Big)\,dx + c_2\int_\Omega |u-g|^2\,dx. \tag{3}$$

Even though the form of these functionals is quite similar to the previous one, the domain of the limiting functional will be different. In fact, as we have $G_\varepsilon(u,1) \leq \int_\Omega |\nabla u|\,dx$, it is clear that the limit of these functionals will be finite if $u \in BV(\Omega)$. In fact we prove (Theorem 4.1 and Example 4.6) that G_ε converge to functionals related to the function-surface energy

$$G(u,K) = |Du|(\Omega \setminus K) + \int_K \frac{|u^+ - u^-|}{1 + |u^+ - u^-|}\,d\mathcal{H}^1 + c_2 \int_{\Omega\setminus K} |u-g|^2\,dx\,, \qquad (4)$$

where $|Du|(A)$ denotes the total variation on A of the distributional derivative Du, and $u^\pm$ are the traces of u on both sides of K. We push this approach further, constructing a variational approximation for a wide class of non-convex functionals defined on spaces of functions of bounded variation.

The paper is divided as follows. In Section 2 we introduce the spaces of generalized functions of bounded variation GBV and $GSBV$, which are needed for a weak formulation of the functionals in (1)–(4), and the notion of Γ-convergence, which precises in which sense the convergence of these functionals is understood. In Section 3 we state the many preliminaries which are needed in the course of the proof. Section 4 is devoted to the statement and proof of the main result, in a slightly more general form than above. The proof of the result lies on a lower bound which is obtained by a new definition of the limit interfacial energy density, taking into account the interaction of the first two integrals of the approximating energies G_ε, and on an upper bound which is obtained by direct construction and a density result of pairs function-polyhedral surface. Section 5 contains the statement and proof of the approximation result for general isotropic functionals with convex bulk energy density and concave surface energy density defined on GBV.

2 Notation

We use standard notation for Sobolev and Lebesgue spaces. $\mathcal{L}^n$ will denote the Lebesgue measure in $\mathbf{R}^n$ and $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure. $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ will be the families of open and Borel sets, respectively. If μ is a Borel measure and E is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. Let $A' \subset\subset A$ be open sets. By a *cut-off function between A' and A* we mean a function $\phi \in C_0^\infty(A)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on A'.

2.1 Generalized functions of bounded variation

Let $u \in L^1(\Omega)$. We say that u is a *function of bounded variation* on Ω if its distributional derivative is a measure; i.e., there exist signed measures μ_{ij} such that

$$\int_\Omega u_i D_j \phi\,dx = -\int_\Omega \phi\,d\mu_{ij}$$

for all $\phi \in C_c^1(\Omega)$. The vector measure $\mu = (\mu_{ij})$ will be denoted by Du. The space of all functions of bounded variation on Ω will be denoted by $BV(\Omega)$.

It can be proven that if $u \in BV(\Omega)$ then the complement of the set of Lebesgue points S_u, that will be called the *jump set* of u, is *rectifiable*, i.e. there exists a countable family (Γ_i) of graphs of Lipschitz functions of $(n-1)$ variables such that $\mathcal{H}^{n-1}(S_u \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$. Hence, a *normal* ν_u can be defined $\mathcal{H}^{n-1}$-a.e. on S_u, as well as the *traces* $u^\pm$ of u on both sides of S_u as

$$u^\pm(x) = \lim_{\rho \to 0+} \fint_{\{y \in B_\rho(x) : \pm\langle y - x, \nu_u(x)\rangle > 0\}} u(y)\, dy\,,$$

where $\fint_B u\, dy = |B|^{-1} \int_B u\, dy\}$.

If $u \in BV(\Omega)$ we define the three measures $D^a u$, $D^j u$ and $D^c u$ as follows. By the Radon Nikodym Theorem we set $Du = D^a u + D^s u$ where $D^a u << \mathcal{L}^n$ and $D^s u$ is the *singular part* of Du with respect to $\mathcal{L}^n$. $D^a u$ is the *absolutely continuous* part of Du with respect to the Lebesgue measure, $D^j u = Du \llcorner S_u$ is the *jump part* of Du, and $D^c u = D^s u \llcorner (\Omega \setminus S_u)$ is the *Cantor part* of Du. We can write then

$$Du = D^a u + D^j u + D^c u\,.$$

It can be seen that $D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u$, and that the Radon Nikodym derivative of Du with respect of $\mathcal{L}^n$ is the *approximate gradient* ∇u of u.

A function $u \in L^1(\Omega)$ is a *special function of bounded variation* on Ω if $D^c u = 0$, or, equivalently, if its distributional derivative can be written as

$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u\,.$$

The space of special functions of bounded variation on Ω is denoted $SBV(\Omega)$. We will also use the auxiliary spaces

$$SBV^p(\Omega) = \{u \in SBV(\Omega) : \ |\nabla u| \in L^p(\Omega),\ \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

We define the space $GBV(\Omega)$ of *generalized functions of bounded variation* as the space of all functions $u \in L^1(\Omega)$ whose truncations $u_T = (-T) \vee (u \wedge T)$ are in $BV(\Omega)$ for any $T > 0$. For such functions we can define $S_u = \bigcup_{T>0} S_{u_T}$, and the approximate gradient and the traces $u^\pm$ as the limits of the corresponding quantities defined for u_T. Moreover, we define the measure $|D^c u| : \mathcal{B}(\Omega) \to [0, +\infty]$ as

$$|D^c u|(B) = \sup_{T>0} |D^c u_T|(B) = \lim_{T \to +\infty} |D^c u_T|(B).$$

If $u \in BV(\Omega)$ $|D^c u|$ coincides with the usual notion of total variation of $D^c u$. Finally, we set

$$GSBV(\Omega = \{u \in GBV(\Omega) : |D^c u| = 0\} = \{u \in L^1(\Omega) : \ u_T \in SBV(\Omega) \text{ for all } T\}.$$

For a detailed study of the properties of BV-functions we refer to [2], [10] and [11]. For an introduction to the study of free-discontinuity problems in the BV setting we refer to [2].

2.2 Relaxation and Γ-convergence

Let (X, d) be a metric space. We first recall the notion of *relaxed functional*. Let $F : X \to \mathbf{R} \cup \{+\infty\}$. Then the relaxed functional $\overline{F}$ of F, or *relaxation* of F, is the greatest d-lower semicontinuous functional less than or equal to F.

We say that a sequence $F_j : X \to [-\infty, +\infty]$ Γ-*converges* to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if for all $u \in X$ we have

(i) (*lower limit inequality*) for every sequence (u_j) converging to u

$$F(u) \leq \liminf_j F_j(u_j); \tag{5}$$

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that

$$F(u) \geq \limsup_j F_j(u_j), \tag{6}$$

or, equivalently by (5),

$$F(u) = \lim_j F_j(u_j). \tag{7}$$

The function F is called the Γ-limit of (F_j) (with respect to d), and we write $F = \Gamma\text{-}\lim_j F_j$. If (F_ε) is a family of functionals indexed by $\varepsilon > 0$ then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}$ for all (ε_j) converging to 0.

The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.1 *Let $F = \Gamma\text{-}\lim_j F_j$, and let a compact set $K \subset X$ exist such that $\inf_X F_j = \inf_K F_j$ for all j. Then*

$$\exists \min_X F = \lim_j \inf_X F_j. \tag{8}$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.

The definition of Γ-convergence can be given pointwise on X. It is convenient to introduce also the notion of Γ-lower and upper limit, as follows: let $F_\varepsilon : X \to [-\infty, +\infty]$ and $u \in X$. We define

$$\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\}; \tag{9}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\}. \tag{10}$$

If $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u)$ then the common value is called the Γ-limit of (F_ε) at u, and is denoted by $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$. Note that this definition is in accord with the previous one, and that F_ε Γ-converges to F if and only if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ at all points $u \in X$.

We recall that:
(i) if $F = \Gamma\text{-}\lim_j F_j$ and G is a continuous function then $F+G = \Gamma\text{-}\lim_j (F_j + G)$;
(ii) the Γ-lower and upper limits define lower semicontinuous functions.

From (i) we get that in the computation of our Γ-limits we can drop all d-continuous terms. Remark (ii) will be used in the proofs combined with approximation arguments.

For an introduction to Γ-convergence we refer to [8]. For an overview of Γ-convergence techniques for the approximation of free-discontinuity problems see [5].

3 Preliminaries

In the following Ω will denote a bounded open set in $\mathbf{R}^n$ with Lipschitz boundary.

We denote by $\mathcal{W}(\Omega)$ the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

(i) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;

(ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite number of pairwise disjoint $(n-1)$-dimensional simplexes;

(iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbf{N}$.

The following result is due to Cortesani [7] (see also [6]).

Theorem 3.1 (Strong approximation in SBV^2) *Let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (w_j) in $\mathcal{W}(\Omega)$ such that $w_j \to u$ strongly in $L^1(\Omega)$, $\nabla w_j \to \nabla u$ strongly in $L^2(\Omega, \mathbf{R}^n)$, $\limsup_{h\to+\infty} \|w_j\|_\infty \le \|u\|_\infty$ and*

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\, d\mathcal{H}^{n-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}$$

for every upper semicontinuous function $\phi : \mathbf{R} \times \mathbf{R} \times S^{n-1} \to [0,+\infty)$ such that $\phi(a,b,\nu) = \phi(b,a,-\nu)$, for every $a, b \in \mathbf{R}$ and $\nu \in S^{n-1}$.

The next result is a particular case of a theorem by Bouchitté, Braides and Buttazzo [4], and deals with relaxation in BV of isotropic functionals.

Theorem 3.2 (Relaxation in BV) *Let $g : \mathbf{R} \to [0,+\infty]$ be a lower semicontinuos function with*

$$g(0) = 0, \qquad \lim_{t\to0+} \frac{g(t)}{t} = 1,$$

and such that the map $t \to g(|t|)$ is subadditive and locally bounded. Let $F : BV(\Omega) \to [0,+\infty]$ be defined by

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|\, dx + \int_{S_u} g(|u^+ - u^-|)\, d\mathcal{H}^{n-1} & \textit{if} \quad u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ +\infty & \textit{otherwise in } BV(\Omega) \end{cases}$$

Then the relaxation of F with respect to the $L^1(\Omega)$-topology is given on $BV(\Omega)$ by the functional

$$\overline{F}(u) = \int_\Omega |\nabla u|\, dx + \int_{S_u} g(|u^+ - u^-|)\, d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

The following lemma is a commonly used tool (see [5]).

Lemma 3.3 (Supremum of measures) *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be an open-set superadditive function, let $\lambda \in \mathcal{M}^+(\Omega)$, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$.*

We finally include a 'slicing' result by Ambrosio (see [1]). We introduce first some notation. Let $\xi \in S^{n-1}$, and let $\Pi_\xi := \{y \in \mathbf{R}^n : \langle y, \xi \rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbf{R}^n$ we define $E_{\xi,y} = \{t \in \mathbf{R} : y + t\xi \in E\}$. Moreover, if $u : \Omega \to \mathbf{R}$ we set $u_{\xi,y} : \Omega_{\xi,y} \to \mathbf{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$.

Theorem 3.4 (a) *Let $u \in BV(\Omega)$. Then, for all $\xi \in S^{n-1}$ the function $u_{\xi,y}$ belongs to $BV(\Omega_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$. For such y we have*

$$u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi \rangle \text{ for a.a. } t \in \Omega_{\xi,y} \tag{11}$$

$$S_{u_{\xi,y}} = \{t \in \mathbf{R} : \; y + t\xi \in S_u\}, \tag{12}$$

$$v(t\pm) = u^\pm(y + t\xi) \qquad \text{or} \qquad v(t\pm) = u^\mp(y + t\xi), \tag{13}$$

according to the cases $\langle \nu_u, \xi \rangle > 0$ or $\langle \nu_u, \xi \rangle < 0$ (the case $\langle \nu_u, \xi \rangle = 0$ being negligible). Moreover, we have

$$\int_{\Pi_\xi} |D^c u_{\xi,y}|(A_{\xi,y}) d\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi \rangle|(A) \tag{14}$$

for all $A \in \mathcal{A}(\Omega)$, and for all Borel functions g

$$\int_{\Pi_\xi} \sum_{t \in S_{u_{\xi,y}}} g(t)\, d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x) |\langle \nu_u, \xi \rangle| d\mathcal{H}^{n-1}. \tag{15}$$

(b) *Conversely, if $u \in L^1(\Omega)$ and for all $\xi \in \{e_1, \ldots, e_n\}$ and for a.a. $y \in \Pi_\xi$ $u_{\xi,y} \in BV(\Omega_{\xi,y})$ and*

$$\int_{\Pi_\xi} |Du_{\xi,y}|(\Omega_{\xi,y})\, d\mathcal{H}^{n-1}(y) < +\infty, \tag{16}$$

then $u \in BV(\Omega)$.

4 The main result

Using the space GBV defined in the previous section, it is possible to give a weak formulation for problems as in (1) and (4), which has been successfully used to obtain solutions of free-discontinuity problems (see [2]). In what follows we drop the term containing $\int |u-g|^2\,dx$, which is of lower order, and does not affect the form of the Γ-limit, and we generalize the form of the functional (3).

Theorem 4.1 *Let $W:[0,1]\to[0,+\infty)$ be a continuous function such that $W(x)=0$ if and only if $x=1$, and let $\psi:[0,1]\to+\infty$ be an increasing lower semicontinuous function with $\psi(0)=0$, $\psi(1)=1$, and $\psi(t)>0$ if $t\neq 0$. Let $G_\varepsilon : L^1(\Omega)\times L^1(\Omega)\to[0,+\infty)$ be defined by*

$$G_\varepsilon(u,v)=\begin{cases}\displaystyle\int_\Omega\Big(\psi(v)|\nabla u|+\frac{1}{\varepsilon}W(v)+\varepsilon|\nabla v|^2\Big)\,dx & \textit{if}\quad u,v\in H^1(\Omega)\\ & \textit{and } 0\le v\le 1 \textit{ a.e.}\\ +\infty & \textit{otherwise.}\end{cases}$$

Then there exists the Γ-$\lim_{\varepsilon\to0+}G_\varepsilon(u,v)=G(u,v)$ with respect to the $L^1(\Omega)\times L^1(\Omega)$-convergence, where

$$G(u,v)=\begin{cases}\displaystyle\int_\Omega|\nabla u|\,dx+\int_{S_u}g(|u^+-u^-|)\,d\mathcal{H}^{n-1}+|D^cu|(\Omega) & \textit{if}\quad u\in GBV(\Omega)\\ & \textit{and } v=1 \textit{ a.e.}\\ +\infty & \textit{otherwise,}\end{cases}$$

and

$$g(z):=\min\{\psi(x)z+2c_W(x):0\le x\le 1\},\tag{17}$$

with $c_W(x):=2\int_x^1\sqrt{W(s)}\,ds$.

The proof of the theorem above will be a consequence of the propositions in the rest of the section . Before entering into the details of the proof, we define also a 'localized version' of our functionals as follows:

$$G_\varepsilon(u,v,A)=\begin{cases}\displaystyle\int_A\Big(\psi(v)|\nabla u|+\frac{1}{\varepsilon}W(v)+\varepsilon|\nabla v|^2\Big)\,dx & \text{if}\quad u,v\in H^1(\Omega)\\ & \text{and } 0\le v\le 1 \text{ a.e.}\\ +\infty & \text{otherwise.}\end{cases}$$

and

$$G(u,v,A)=\begin{cases}\displaystyle\int_A|\nabla u|\,dx+\int_{S_u\cap A}g(|u^+-u^-|)\,d\mathcal{H}^{n-1}+|D^cu|(A) & \\ & \text{if } u\in GBV(\Omega) \text{ and } v=1 \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}$$

for any $A \Subset \Omega$ bounded open set.

Remark 4.2 By the assumptions on ψ and W, it can be easily proved that g satisfies the following properties

(i) g is increasing, $g(0) = 0$ and

$$\lim_{z \to +\infty} g(z) = 2c_W(0) = 4 \int_0^1 \sqrt{W(s)}\, ds;$$

(ii) g is subadditive, i.e.

$$g(z_1 + z_2) \leq g(z_1) + g(z_2) \qquad \forall z_1,\ z_2 \in \mathbf{R}^+;$$

(iii) g is Lipschitz-continuous with Lipschitz constant 1;

(iv) $g(z) \leq z$ for all $z \in \mathbf{R}^+$ and

$$\lim_{z \to 0^+} \frac{g(z)}{z} = 1;$$

(v) for any $T > 0$ there exists a constant $c_T > 0$ such that $z \leq c_T\, g(z)$ for all $z \in [0, T]$.

Proposition 4.3 *Let $n = 1$. Then $G(u,v) \leq \Gamma\text{-}\liminf_{\varepsilon \to 0^+} G_\varepsilon(u,v)$ for all $u, v \in L^1(\Omega)$.*

PROOF. It suffices to consider the case in which the right-hand side is finite. Let $\varepsilon_j \to 0^+$, $u_j \to u$ and $v_j \to v$ in $L^1(\Omega)$ be such that $\lim_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j) = \Gamma\text{-}\liminf_{\varepsilon \to 0^+} G_\varepsilon(u,v)$. Up to passing to subsequences we may suppose

$$u_j \to u, \text{ and } v_j \to v \text{ a.e.} \tag{18}$$

We have

$$\int_\Omega W(v_j)\, dx < c\varepsilon_j;$$

hence, by the continuity of W, for any $\eta > 0$ $\mathcal{L}^1\big(\{x \in \Omega : W(v(x)) > \eta\}\big) = \lim_{j \to +\infty} \mathcal{L}^1\big(\{x \in \Omega : W(v_j(x)) > \eta\}\big) = 0$. We conclude that $W(v) = 0$ a.e., i.e. $v = 1$ a.e.

By simplicity, suppose that $\Omega = (a,b)$ (otherwise we split Ω into its connected components). We now use a discretization argument similar to the one used in the proof of [3]. Let $N \in \mathbf{N}$ and consider the intervals

$$I_N^k = \Big(a + \frac{(k-1)}{N}(b-a), a + \frac{k}{N}(b-a)\Big), \qquad k \in \{1, .., N\}.$$

Up to passing to subsequences we may suppose that

$$\lim_{j\to+\infty} \inf_{I_N^k} v_j$$

exists for all $N \in \mathbf{N}$ and $k \in \{1,..,N\}$. Let $z \in (0,1)$ be fixed and consider the set

$$J_N^z = \Big\{k \in \{1,..,N\} : \lim_{j\to+\infty} \inf_{I_N^k} v_j \leq z\Big\}.$$

Note that for any (α,β) interval in $\mathbf{R}$ and for any $w \in H^1(\alpha,\beta)$ we have, by Young's inequality,

$$\int_\alpha^\beta \Big(\frac{1}{\varepsilon}W(w) + \varepsilon|w'|^2\Big)\,dx \geq 2\int_\alpha^\beta \sqrt{W(w)}|w'|\,dx \geq 2\Big|\int_{w(\alpha)}^{w(\beta)} \sqrt{W(s)}\,ds\Big|.$$

From this inequality we deduce, arguing as in [3], that

$$\Big(2\int_z^1 \sqrt{W(s)}\,ds\Big)\#J_N^z \leq \lim_{j\to+\infty} G_{\varepsilon_j}(u_j,v_j) < +\infty.$$

Then

$$\#J_N^z \leq C$$

with C independent of N. Hence, up to a subsequence, we may suppose

$$J_N^z = \{k_1^N,..,k_L^N\}$$

with L independent of N, and up to a further subsequence that there exist $S = \{t_1,..,t_L\} \subset [a,b]$ such that

$$\lim_{N\to+\infty} \frac{k_i^N}{N} = t_i$$

for any $i \in \{1,..,L\}$. For every $\eta > 0$ we have

$$I_N^k \subset S_\eta := S + [-\eta,\eta]$$

for all $k \in J_N^z$ and for N large enough. Then

$$\begin{aligned}
\liminf_{j\to+\infty} G_{\varepsilon_j}(u_j,v_j) &\geq \liminf_{j\to+\infty} G_{\varepsilon_j}(u_j,v_j,\Omega\setminus S_\eta) \\
&\quad + \liminf_{j\to+\infty} \sum_{i=1}^L G_{\varepsilon_j}(u_j,v_j,(t_i-\eta,t_i+\eta)) \\
&\geq \liminf_{j\to+\infty} \psi(z)\int_{\Omega\setminus S_\eta} |u_j'|\,dt \\
&\quad + \sum_{i=1}^L \liminf_{j\to+\infty} G_{\varepsilon_j}(u_j,v_j,(t_i-\eta,t_i+\eta)). \qquad (19)
\end{aligned}$$

With fixed $i \in \{1, \ldots, L\}$, we focus our attention on the term $G_{\varepsilon_j}(u_j, v_j, (t_i - \eta, t_i + \eta))$. By definition and by (18), we have that for any $\delta > 0$ there exist $x_1,\ x_2 \in (t_i - \eta, t_i + \eta)$ such that

$$\begin{aligned} \lim_{j \to +\infty} u_j(x_1) &= u(x_1) < \operatorname*{ess-\,inf}_{(t_i-\eta, t_i+\eta)} u + \delta, \\ \lim_{j \to +\infty} u_j(x_2) &= u(x_2) > \operatorname*{ess-\,sup}_{(t_i-\eta, t_i+\eta)} u - \delta, \\ \lim_{j \to +\infty} v_j(x_1) &= \lim_{j \to +\infty} v_j(x_2) = 1. \end{aligned} \tag{20}$$

Let $x_j^i \in [x_1, x_2]$ be such that $v_j(x_j^i) = \inf_{[x_1,x_2]} v_j$. Then we obtain the following estimate:

$$\begin{aligned} G_{\varepsilon_j}(u_j, v_j, I_N^{k_i}) &\geq G_{\varepsilon_j}(u_j, v_j, (x_1, x_2)) \\ &\geq \psi(v_j(x_j^i))\Big|\int_{x_1}^{x_2} u_j'\,dx\Big| + 2\int_{x_1}^{x_2} \sqrt{W(v_j)}|v_j'|\,dx \\ &\geq \psi(v_j(x_j^i))|u_j(x_2) - u_j(x_1)| \\ &\qquad + 2\int_{v_j(x_j^i)}^{v_j(x_1)} \sqrt{W(s)}\,ds + 2\int_{v_j(x_j^i)}^{v_j(x_2)} \sqrt{W(s)}\,ds \\ &\geq \inf_{t \in [0,1]} \Big\{\psi(t)|u_j(x_2) - u_j(x_1)| \\ &\qquad + 2\Big(\int_t^{v_j(x_1)} \sqrt{W(s)}\,ds + \int_t^{v_j(x_2)} \sqrt{W(s)}\,ds\Big)\Big\}. \end{aligned} \tag{21}$$

Letting $j \to +\infty$ and taking into account (20), we get

$$\begin{aligned} &\liminf_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j, I_N^{k_i} \\ \geq\ &\inf_{t \in [0,1]} \Big\{\psi(t)\Big|\operatorname*{ess-\,sup}_{(t_i-\eta, t_i+\eta)} u - \operatorname*{ess-\,inf}_{(t_i-\eta, t_i+\eta)} u - 2\delta\Big| + 4\int_t^1 \sqrt{W(s)}\,ds\Big\}. \end{aligned}$$

Thus, by the arbitrariness of $\delta > 0$,

$$\liminf_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j, (t_i - \eta, t_i + \eta)) \geq g\Big(\operatorname*{ess-\,sup}_{(t_i-\eta, t_i+\eta)} u - \operatorname*{ess-\,inf}_{(t_i-\eta, t_i+\eta)} u\Big) \tag{22}$$

Now we turn back to the estimate (19). Since $\sup_j G_{\varepsilon_j}(u_j, v_j) < +\infty$, by (19) we get the equiboundness of $\int_{\Omega \setminus S_\eta} |u_j'|\,dt$. Hence $u \in BV(\Omega \setminus S_\eta)$ and, by (19) and (22),

$$\liminf_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j) \geq \psi(z)|Du|(\Omega \setminus S_\eta) + \sum_{i=1}^{L} g\Big(\operatorname*{ess-\,sup}_{(t_i-\eta, t_i+\eta)} u - \operatorname*{ess-\,inf}_{(t_i-\eta, t_i+\eta)} u\Big). \tag{23}$$

By the arbitrariness of η, we deduce that $u \in BV(\Omega \setminus S)$, i.e., since S is finite, $u \in BV(\Omega)$. Then, letting $\eta \to 0$ in (23), we get

$$\liminf_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j) \geq \psi(z)|Du|(\Omega \setminus S) + \sum_{i=1}^{L} g(|u^+ - u^-|(t_i))$$

$$\geq \psi(z)|Du|(\Omega \setminus S_u) + \sum_{t \in S_u} \Big(g(|u^+ - u^-|(t)) \wedge \psi(z)|u^+ - u^-|(t) \Big). \tag{24}$$

Finally, letting $z \to 1$ in (24) we obtain the required inequality, since $g(t) \leq t$. □

We recover, now, the n-dimensional analogue of the previous inequality, by using Theorem 3.4.

Proposition 4.4 *Let $n \in \mathbf{N}$. Then $G(u,v) \leq \Gamma\text{-}\liminf_{\varepsilon \to 0^+} G_\varepsilon(u,v)$ for all $u, v \in L^1(\Omega)$.*

Proof. In the following we will use the notation $G' = \Gamma\text{-}\liminf_{\varepsilon \to 0^+} G_\varepsilon$.

Let $\xi \in S^{n-1}$ be fixed and let π_ξ be the hyperplane through 0 orthogonal to ξ. For any $u \in L^1(\Omega)$, $A \in \mathcal{A}(\Omega)$, $y \in \pi_\xi$ we set

$$A_{\xi y} := \{t \in \mathbf{R} : y + t\xi \in A\}, \quad u_{\xi y}(t) := u(y + t\xi).$$

In particular, if $u \in H^1(\Omega)$, we get

$$u'_{\xi y}(t) := \langle \nabla u(y + t\xi), \xi \rangle.$$

For any $u, v \in H^1(\Omega)$, $0 \leq v \leq 1$, we have, by Fubini's Theorem,

$$\begin{aligned}
& G_\varepsilon(u, v, A) \\
= & \int_{\pi_\xi} \int_{A_{\xi y}} \Big(\psi(v(y + t\xi)) |\nabla u(y + t\xi)| \\
& \quad + \frac{1}{\varepsilon} W(v(y + t\xi)) + \varepsilon |\nabla v(y + t\xi)|^2 \Big) \, dt \, d\mathcal{H}^{n-1}(y) \\
\geq & \int_{\pi_\xi} \int_{A_{\xi y}} \Big(\psi(v_{\xi y}(t)) |u'_{\xi y}| + \frac{1}{\varepsilon} W(v_{\xi y}(t)) + \varepsilon |v'_{\xi y}(t)|^2 \Big) \, dt \, d\mathcal{H}^{n-1}(y) \\
= & \int_{\pi_\xi} \mathcal{G}_\varepsilon(u_{\xi y}, v_{\xi y}, A_{\xi y}) \, d\mathcal{H}^{n-1}(y),
\end{aligned} \tag{25}$$

where $\mathcal{G}_\varepsilon$ is defined by

$$\mathcal{G}_\varepsilon(u, v, I) = \begin{cases} \displaystyle\int_I \Big(\psi(v)|u'| + \frac{1}{\varepsilon} W(v) + \varepsilon |v''|^2 \Big) \, dt & \text{if } u, v \in H^1(I) \\ & \text{and } 0 \leq v \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

for any $u, v \in L^1(I)$ and $I \subset \mathbf{R}$ open and bounded.

Let $\varepsilon_j \to 0$ and let $u_j \to u$, $v_j \to v$ in $L^1(\Omega)$ be such that

$$\liminf_{j\to+\infty} G_{\varepsilon_j}(u_j, v_j) \leq +\infty. \tag{26}$$

Then $u_j,\ v_j \in H^1(\Omega),\ 0 \leq v_j \leq 1$ a.e. and, as in the proof of Proposition 4.3, $v = 1$ a.e. Moreover, by Fubini's Theorem, $(u_j)_{\xi y} \to u_{\xi y}, (v_j)_{\xi y} \to 1$ in $L^1(\Omega_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \pi_\xi$.

Thus by Proposition 4.3 and by Fatou's Lemma we get

$$\begin{aligned}
&\liminf_{j\to+\infty} G_{\varepsilon_j}(u_j, v_j, A) \\
\geq\ & \int_{\pi_\xi} \liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}((u_j)_{\xi y}, (v_j)_{\xi y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y) \\
\geq\ & \int_{\pi_\xi} \Big(\int_{A_{\xi y}} |u'_{\xi y}|\, dt + \int_{S_{u_{\xi y}}\cap A_{\xi y}} g(|u^+_{\xi y} - u^-_{\xi y}|)\, d\# + |D^c u_{\xi y}|(A_{\xi y})\Big)\, d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{27}$$

Let $T > 0$ and set

$$u_T = (-T) \vee (u \wedge T).$$

Since g is increasing, it is clear that we decrease the last term in (27) if we substitute u by u_T. Moreover, since $u_T \in L^\infty(\Omega)$, with $\|u_T\|_\infty \leq T$, by Remark 4.2(v), we have

$$|u^+_T - u^-_T| \leq c_T\, g(|u^+_T - u^-_T|)$$

for a suitable constant c_T depending only on T. Then, by (26) and (27), we have

$$\int_{\pi_\xi} |Du_T|(A_{\xi y})\, d\mathcal{H}^{n-1}(y) < +\infty.$$

Thus, applying Theorem 3.4, we get that $u_T \in BV(\Omega)$ and, by the arbitrariness of (u_j) and(v_j),

$$G'(u,1,A) \geq \int_A |\langle \nabla u_T, \xi\rangle|\, dx + \int_{S_u\cap A} g(|u^+_T - u^-_T|)|\langle \nu_u, \xi\rangle|\, d\mathcal{H}^{n-1} + |\langle D^c u_T, \xi\rangle|(A) \tag{28}$$

for all $A \in \mathcal{A}(\Omega)$ and $\xi \in S^{n-1}$.

Consider the superadditive increasing function defined on $\mathcal{A}(\Omega)$ by

$$\gamma(A) := G'(u, 1, A)$$

and the Radon measure

$$\lambda := \mathcal{L}^n \llcorner \Omega + g(|u^+_T - u^-_T|)\mathcal{H}^{n-1} \llcorner S_{u_T} + |D^c u_T|.$$

Fixed a sequence $(\xi_i)_{i\in\mathbf{N}}$, dense in S^{n-1}, we have, by (28),

$$\gamma(A) \geq \int_A \psi_i \, d\lambda$$

for all $i \in \mathbf{N}$, where

$$\psi_i(x) = \begin{cases} |\langle \nabla u_T(x), \xi_i \rangle| & \mathcal{L}^n \text{ a.e. on } \Omega \\ |\langle \nu_u(x), \xi_i \rangle| & |D^c u_T| \text{ a.e. on } \Omega \setminus S_{u_T} \\ |\langle \nu_u(x), \xi_i \rangle| & \mathcal{H}^{n-1} \text{ a.e. on } S_{u_T}. \end{cases}$$

Hence, applying Lemma 3.3, we get

$$G'(u,1,A) \geq \int_A |\nabla u_T| \, dx + \int_{S_{u_T}\cap A} g(|u_T^+ - u_T^-|) \, d\mathcal{H}^{n-1} + |D^c u_T|(A) \tag{29}$$

for all $A \in \mathcal{A}(\Omega)$. In particular

$$G'(u,1,\Omega) \geq \int_\Omega |\nabla u_T| \, dx + \int_{S_{u_T}} g(|u_T^+ - u_T^-|) \, d\mathcal{H}^{n-1} + |D^c u_T|(\Omega). \tag{30}$$

Finally, by the arbitrariness of $T > 0$, $u \in GBV(\Omega)$ and the thesis follows letting $T \to +\infty$ in (30). □

Proposition 4.5 *We have* $\Gamma\text{-}\limsup_{\varepsilon\to 0^+} G_\varepsilon(u,v) \leq G(u,v)$ *for all* $u, v \in L^1(\Omega)$.

PROOF. It suffices to prove the inequality for $v = 1$ a.e. Since we will use density and relaxation arguments, we divide the proof into five steps, passing from a particular choice of u to the general one. In the following we will use the notation $G'' = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} G_\varepsilon$.

Step 1. Suppose that $u \in \mathcal{W}(\Omega)$ and

$$\overline{S}_u = \Omega \cap K$$

with K a $(n-1)$-dimensional simplex. Up to a translation and rotation argument, we can suppose that K is contained in the hyperplane $\pi := \{x_n = 0\}$. Set

$$h(y) := u^+(y) - u^-(y), \quad y \in \overline{S}_u.$$

By our hypotheses on u, h is regular on $\overline{S}_u$; hence, fixed $\delta > 0$, we can find a triangulation $\{T_i\}_{i=1}^N$ of $\overline{S}_u$ such that

$$|h(y_1) - h(y_2)| < \delta \quad \text{if } y_1, y_2 \in T_i.$$

Let $h_\delta : \overline{S}_u \to \mathbf{R}$ be defined as

$$h_\delta(y) := z_i \quad y \in T_i,$$

where $z_i := \min\{h(y) : \ y \in \overline{T}_i\}$. Since $\|h - h_\delta\|_\infty < \delta$, by Remark 4.2 (iii), we have that

$$\int_{S_u} g(h_\delta(y))\, d\mathcal{H}^{n-1} \leq \int_{S_u} g(h(y))\, d\mathcal{H}^{n-1} + \delta\mathcal{H}^{n-1}(\overline{S}_u).$$

Let x_{z_i} realize the minimum in (17) for $z = z_i$. Fixed $\eta > 0$, there exists $T(\eta) > 0$ such that

$$\min\Big\{\int_0^T \Big(|v'|^2 + W(v)\Big)\, dt : v \in H^1(0,T),\ v(0) = x_{z_i},\ v(T) = 1\Big\} \leq c_W(x_{z_i}) + \eta \tag{31}$$

for all $T \geq T(\eta)$ and for any $i = 1, .., N$. Let $v(z_i, \cdot)$ realize the minimum in (31).

For $r > 0, \varepsilon > 0$ and $i \in \{1, .., N\}$, set

$$B_r := \Big\{(y,t) \in \Omega :\ y \in \overline{S}_u,\ |t| < r\Big\} \quad \text{and} \quad T_i^\varepsilon := \Big\{y \in T_i :\ d(y, \partial T_i) > \varepsilon\Big\},$$

and let $\phi_\varepsilon^i : \mathbf{R}^{(n-1)} \to \mathbf{R}$ be a cut-off function between T_i^ε and T_i such that $\|\nabla \phi_\varepsilon^i\|_\infty < C\varepsilon^{-1}$. Fix a sequence (ξ_ε) such that $\lim_{\varepsilon\to 0+} \frac{\xi_\varepsilon}{\varepsilon} = 0$, set $T_\varepsilon := T(\eta)\varepsilon + \xi_\varepsilon$, and define

$$v_\varepsilon(y,t) := \begin{cases} 1 & \text{if} \quad (y,t) \in \Omega \setminus B_{T_\varepsilon} \\ \phi_\varepsilon^i(y) v_\varepsilon^i(t) + (1 - \phi_\varepsilon^i(y)) & \text{if} \quad y \in T_i,\ |t| < T_\varepsilon, \end{cases}$$

where

$$v_\varepsilon^i(t) := \begin{cases} x_{z_i} & \text{if} \quad |t| < \xi_\varepsilon \\ v\Big(z_i, \frac{|t| - \xi_\varepsilon}{\varepsilon}\Big) & \text{if} \quad \xi_\varepsilon < |t| < T_\varepsilon. \end{cases}$$

We have that $(v_\varepsilon) \in H^1(\Omega)$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$ as $\varepsilon \to 0+$. Hence, we get

$$\begin{aligned} & \int_\Omega \Big(\varepsilon|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon)\Big)\, dx \qquad (32) \\ = \ & \sum_{i=1}^N \int_{T_i^\varepsilon} 2\int_{\xi_\varepsilon}^{T_\varepsilon} \frac{1}{\varepsilon}\Big(\Big|v'\Big(z_i, \frac{|t| - \xi_\varepsilon}{\varepsilon}\Big)\Big|^2 + W\Big(v\Big(z_i, \frac{|t| - \xi_\varepsilon}{\varepsilon}\Big)\Big)\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\ & + \sum_{i=1}^N \int_{T_i} \int_{-\xi_\varepsilon}^{\xi_\varepsilon} \Big(\varepsilon|\nabla \phi_\varepsilon^i(y)|^2 |z_i - 1|^2 + \frac{1}{\varepsilon}W(v_\varepsilon(y,t))\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\ & + \sum_{i=1}^N \int_{T_i \setminus T_i^\varepsilon} \int_{\xi_\varepsilon}^{T_\varepsilon} \Big(\varepsilon|\nabla \phi_\varepsilon^i(y)|^2 \Big|v\Big(z_i, \frac{|t| - \xi_\varepsilon}{\varepsilon}\Big) - 1\Big|^2 \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{\varepsilon}|\phi^i_\varepsilon(y)|^2\Big|v'\Big(z_i,\frac{|t|-\xi_\varepsilon}{\varepsilon}\Big)\Big|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y) \\
& +\sum_{i=1}^N\int_{T_i\setminus T_i^\varepsilon}\int_{\xi_\varepsilon}^{T_\varepsilon}\frac{1}{\varepsilon}W(v_\varepsilon(y,t))\,dt\,d\mathcal{H}^{n-1}(y) \\
\leq\ & \sum_{i=1}^N\int_{T_i^\varepsilon}2\int_0^T\Big(|v'(z_i,t)|^2+W(v(z_i,t))\Big)\,dt\,d\mathcal{H}^{n-1}(y) \\
& +c\frac{\xi_\varepsilon}{\varepsilon}\mathcal{H}^{n-1}(S_u)+c(\eta)\sum_{i=1}^N\mathcal{H}^{n-1}(T_i\setminus T_i^\varepsilon) \\
\leq\ & \sum_{i=1}^N 2\int_{T_i}c_W(x_{z_i})\,d\mathcal{H}^{n-1}(y)+2\eta\mathcal{H}^{n-1}(S_u)+O(\varepsilon).
\end{aligned}
$$

We now construct a recovery sequence u_ε. Let

$$
\tilde{u}_\varepsilon(z_1,z_2,t)=\begin{cases} z_1 & -T_\varepsilon<t<-\xi_\varepsilon \\ \frac{z_2-z_1}{2\xi_\varepsilon}(t+\xi_\varepsilon)+z_1 & |t|<\xi_\varepsilon \\ z_2 & \xi_\varepsilon<t<T_\varepsilon \end{cases}
$$

and set

$$
u_\varepsilon(y,t)=\begin{cases} u(y,t) & |t|>T_\varepsilon \\ \tilde{u}_\varepsilon\Big(u(y,-T_\varepsilon),u(y,T_\varepsilon),t\Big) & |t|<T_\varepsilon\ . \end{cases}
$$

It can be easily verified that $u_\varepsilon\in H^1(\Omega)$ and $u_\varepsilon\to u$ in $L^1(\Omega)$ as $\varepsilon\to 0^+$. Moreover, we have

$$
\begin{aligned}
\int_\Omega\psi(v_\varepsilon)|\nabla u_\varepsilon|\,dx\ \leq\ & \sum_{i=1}^N\int_{T_i^\varepsilon}\int_{-\xi_\varepsilon}^{\xi_\varepsilon}\frac{1}{2\xi_\varepsilon}\psi(x_{z_i})|u(y,T_\varepsilon)-u(y,-T_\varepsilon)|\,dt\,d\mathcal{H}^{n-1}(y) \\
& +\int_{\Omega\setminus B_{t_\varepsilon}}|\nabla u|\,dx+c\mathcal{H}^{n-1}(T_i\setminus T_i^\varepsilon)+O(\varepsilon) \\
=\ & \int_\Omega|\nabla u|\,dx+\sum_{i=1}^N\int_{T_i}\psi(x_{z_i})|u^+-u^-|(y)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon).
\end{aligned}
\tag{33}
$$

Letting, now, ε tend to 0^+, we obtain, by (32) and (33),

$$
\begin{aligned}
& G''(u,1)\leq\limsup_{\varepsilon\to 0^+}G_\varepsilon(u_\varepsilon,v_\varepsilon) \\
\leq\ & \int_\Omega|\nabla u|\,dx+\sum_{i=1}^N\int_{T_i}(|u^+-u^-|(y)\psi(x_{z_i})+2c_W(x_{z_i}))\,d\mathcal{H}^{n-1}(y)+c\eta
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_\Omega |\nabla u|\,dx + \sum_{i=1}^N \int_{T_i} (z_i \psi(x_{z_i}) + 2c_W(x_{z_i}))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta) \\
&= \int_\Omega |\nabla u|\,dx + \int_{S_u} g(h_\delta(y))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta) \\
&\leq \int_\Omega |\nabla u|\,dx + \int_{S_u} g(|u^+ - u^-|(y))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta).
\end{aligned}
$$

Letting η and δ tend to 0^+, we obtain the required inequality.

In order to use the same costruction as above in the case $\overline{S}_u = \Omega \cap \left(\bigcup_{i=1}^M K_i\right)$, with $M > 1$, we now show that we can replace (u_ε) by a new sequence $(\hat{u}_\varepsilon)$ such that $\hat{u}_\varepsilon \neq u$ only in a small neighbourhood of K. To this end we again use a cut-off argument. Set

$$K_\varepsilon := \{y \in \pi : d(y,K) < \varepsilon\}$$

and let $\phi_\varepsilon : \mathbf{R}^{n-1} \to \mathbf{R}$ be a cut-off function between K and K_ε with $|\nabla\phi_\varepsilon|_\infty \leq c\varepsilon^{-1}$. Define

$$\hat{u}_\varepsilon(y,t) := \phi_\varepsilon(y)u_\varepsilon(y,t) + (1-\phi_\varepsilon(y))u(y,t) \qquad (y,t) \in \Omega.$$

We have

$$
\begin{aligned}
\hat{u}_\varepsilon(y,t) = u_\varepsilon(y,t) \qquad &\text{if } (y,t) \in B_{T_\varepsilon}, \\
\hat{u}_\varepsilon(y,t) = u(y,t) \qquad &\text{if } (y,t) \in \Omega \setminus K_\varepsilon \times (-T_\varepsilon, T_\varepsilon).
\end{aligned}
\tag{34}
$$

Then

$$
\begin{aligned}
\int_{\Omega\setminus B_{T_\varepsilon}} |\nabla \hat{u}_\varepsilon|\,dx \;&\leq\; \int_{\Omega\setminus K_\varepsilon\times(-T_\varepsilon,T_\varepsilon)} |\nabla u|\,dx \\
&\quad + \int_{\Omega\cap(K_\varepsilon\setminus K)} \int_{-T_\varepsilon}^{T_\varepsilon} \Big(|\nabla\phi_\varepsilon(y)||u_\varepsilon(y,t) - u(y,t)|\Big)\,dt\,d\mathcal{H}^{n-1}(y) \\
&\quad + \int_{\Omega\cap(K_\varepsilon\setminus K)} \int_{-T_\varepsilon}^{T_\varepsilon} \Big(\phi_\varepsilon(y)|\nabla u_\varepsilon(y,t)| \\
&\qquad\qquad + (1-\phi_\varepsilon(y))|\nabla u(y,t)|\Big)\,dt\,d\mathcal{H}^{n-1}(y) \\
&\leq \int_\Omega |\nabla u|\,dx + c\frac{T_\varepsilon}{\varepsilon}\mathcal{H}^{n-1}(K_\varepsilon \setminus K) + O(\varepsilon).
\end{aligned}
$$

Thus

$$\limsup_{\varepsilon\to 0^+} \int_{\Omega\setminus B_{T_\varepsilon}} |\nabla\hat{u}_\varepsilon|\,dx = \int_\Omega |\nabla u|\,dx,$$

and, by (34), we still have

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon(\hat{u}_\varepsilon, v_\varepsilon) \leq G(u,1) + c(\eta+\delta).$$

Step 2. If $u \in \mathcal{W}(\Omega)$ with $\overline{S}_u = \Omega \cap \left(\bigcup_{i=1}^M K_i\right)$, we can generalize in a very natural way the construction of the recovery sequences $\hat{u}_\varepsilon$ and v_ε in Step 1, since this construction modifies u and v only in a small neighbourhood of each sets K_i.

Step 3. Let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Then, applying Theorem 3.1 with $\phi(a, b, \nu) = g(|a - b|)$, there exists a sequence $(w_j) \in \mathcal{W}(\Omega)$ such that

$$w_j \to u \text{ in } L^1(\Omega), \text{ and } \limsup_{j \to +\infty} G(w_j, 1) \leq G(u, 1).$$

Then, by the previous steps and by the lower semicontinuity of G''

$$G''(u, 1) \leq \liminf_{j \to +\infty} G''(w_j, 1) \leq \liminf_{j \to +\infty} G(w_j, 1) \leq G(u, 1).$$

Step 4. Since g satisfies the hypotheses of Theorem 3.2, the relaxation with respect to $L^1(\Omega)$-topology of the functional

$$F(u) := \begin{cases} G(u, 1) & \text{if} \quad u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ \\ +\infty & \text{otherwise in } BV(\Omega) \end{cases}$$

is given by

$$\overline{F}(u) = G(u, 1)$$

for all $u \in BV(\Omega)$. Then by the previous steps and by the lower semicontinuity of G'' we get

$$G''(u, 1) \leq \overline{F}(u) = G(u, 1)$$

for any $u \in BV(\Omega)$.

Step 5. We recover the general case by a truncation argument. Let $u \in GBV(\Omega)$ and let $u_j = (-hj) \vee (u \wedge j)$. Then

$$\lim_{j \to +\infty} G(u_j, 1) = G(u, 1).$$

Since $u_j \to u$ in $L^1(\Omega)$ we get the thesis by the lower semicontinuity of G''. □

Example 4.6 Let $W(v) = (1 - v)^2/4$, so that $c_W(z) = (1 - z)^2/2$. We then have

(a) if $\psi(v) = v^2$ then $g(z) = |z|/(1 + |z|)$;

(b) if $\psi(v) = v$ then $g(z) = \begin{cases} |z| - (z^2/4) & \text{if } |z| \leq 2 \\ 1 & \text{if } |z| > 2; \end{cases}$

(c) if $\psi(v) = \begin{cases} 0 & \text{if } v = 0 \\ 1 & \text{otherwise,} \end{cases}$ then $g(z) = \min\{|z|, 1\}$.

Note that in the first case we always have interaction between the bulk term and the 'surface term' of G_ε (i.e. $x \neq 1$ in the definition of g) contrary to what happens in the Ambrosio Tortorelli approach. The interaction also occurs in the second case for $|z| < 2$. Note moreover that in the third case the minimal x in the definition of $g(z)$ does not vary with continuity at $z = 0$.

5 Approximation of general functionals

In this section we show how Theorem 4.1 can be used to obtain an approximation of general (isotropic) energies defined on $GSBV$.

Proposition 5.1 *Let Ω be a bounded open subset of $\mathbf{R}^n$, let W and ψ be defined as in Theorem 4.1, let $f : [0,+\infty) \to [0,+\infty)$ be a convex function with minimum in 0 satisfying*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 1, \tag{35}$$

and let $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty)$ be defined by

$$G_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi(v) f(|\nabla u|) + \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2\Big)\, dx & \text{if } u, v \in H^1(\Omega) \\ & \text{and } 0 \le v \le 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the Γ-$\lim_{\varepsilon\to 0+} G_\varepsilon(u,v) = G(u,v)$ with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence, where

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega f(|\nabla u|) dx + \int_{S_u} g(|u^+ - u^-|) d\mathcal{H}^{n-1} + |D^c u|(\Omega) & \text{if } u \in GBV(\Omega) \\ & \text{and } v = 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

and g is defined in (17).

Proof. The estimate for the Γ-lim inf can be performed as in Proposition 4.3, noting that in (21) we obtain, by Jensen's inequality,

$$G_{\varepsilon_j}(u_j, v_j, I_N^{k-i}) \ge \psi(v_j(x_j^i)) |x_2 - x_1| f\Big(\frac{u(x_2) - u(x_1)}{x_2 - x_1}\Big) + 2\int_{x_1}^{x_2} \sqrt{W(v_j)}|v_j'|\, dx,$$

from which the lower bound can be easily obtained taking into account (35). The rest of the proof can be obtained following Propositions 4.4 and 4.5. □

Remark 5.2 Let $K > 0$ and $N \ge 2$, let

$$0 = a_0 < a_1 < \cdots < a_N = 1, \qquad 0 = b_N < b_{N-1} \cdots < b_0 = K,$$

and let f and W be as in the previous proposition. Then there exists ψ satisfying the hypotheses in Theorem 4.1 such that, if $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty)$ is defined by

$$G_\varepsilon(u,v)=\begin{cases}\displaystyle\int_\Omega\Big(\psi(v)f(|\nabla u|)+\frac{K}{\varepsilon}W(v)+\varepsilon K|\nabla v|^2\Big)\,dx & \text{if } \ u,v\in H^1(\Omega)\\ & \text{and } 0\le v\le 1 \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}$$

then the thesis of the previous proposition holds with $g:[0,+\infty)\to[0,+\infty)$ given by

$$g(z)=\min\{a_i z+b_i\}.$$

In fact, in this case the formula for g can be easily inverted, obtaining ψ as the piecewise constant function given by $\psi(0)=0$ and

$$\psi(\xi)=a_i \qquad \text{if } c_W^{-1}(b_{i-1}/2)<\xi\le c_W^{-1}l(b_i/2),$$

where c_W is defined in Theorem 4.1.

Proposition 5.3 *Let W be as in Theorem 4.1. Let $\varphi,\vartheta:[0,+\infty)\to[0,+\infty)$ be functions satisfying*

(i) *φ is convex and even, $\lim_{t\to+\infty}\varphi(t)/t=+\infty$;*

(ii) *ϑ is concave and even, $\lim_{t\to0^+}\vartheta(t)/t=+\infty$.*

Then there exist two sequences of functions (φ_j) and (ψ_j), and two sequences of positive real numbers (k_j) and (ε_j), converging to $\sup\vartheta$ and 0, respectively, such that if we define

$$G_j(u,v)=\begin{cases}\displaystyle\int_\Omega\Big(\psi_j(v)\varphi_j(|\nabla u|)+\frac{k_j}{\varepsilon_j}W(v)+k_j\varepsilon_j|\nabla v|^2\Big)\,dx & \textit{if } u,v\in H^1(\Omega)\\ & \textit{and } 0\le v\le 1 \textit{ a.e.}\\ +\infty & \textit{otherwise,}\end{cases}$$

then there exists the Γ-$\lim_{j\to+\infty}G_j(u,v)=G(u,v)$ with respect to the $L^1(\Omega)\times L^1(\Omega)$-convergence, where

$$G(u,v)=\begin{cases}\displaystyle\int_\Omega\varphi(|\nabla u|)\,dx+\int_{S_u}\vartheta(u^+-u^-)d\mathcal{H}^{n-1} & \textit{if } u\in GSBV(\Omega)\\ & \textit{and } v=1 \textit{ a.e.}\\ +\infty & \textit{otherwise.}\end{cases}$$

PROOF. Let $\vartheta_j:[0,+\infty)\to[0,+\infty)$ be functions of the form

$$\vartheta_j(z)=\min\{A_i^j z+B_i^j\},$$

with $0=A_0^j<\cdots<A_j^j=j$ converging increasingly to ϑ, and let $\varphi_j:[0,+\infty)\to[0,+\infty)$ be convex even functions with

$$\lim_{t\to+\infty}\frac{\varphi_j(t)}{t}=j,$$

converging increasingly to φ. Let $k_j = \max \varphi_j$.

Set $g_j = \vartheta_j/j$, $K_j = k_j/j$ and $f_j = \varphi_j/j$. By the previous remark, applied with $g = g_j$, $f = f_j$ and $K = K_j$, we can find $\psi =: \psi_j$ such that if we let $G^j_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty)$ be defined by

$$G^j_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi_j(v)\varphi_j(|\nabla u|) + \frac{k_j}{\varepsilon} W(v) + \varepsilon k_j |\nabla v|^2\Big)\, dx & \text{if} \quad u, v \in H^1(\Omega) \\ & \text{and } 0 \le v \le 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

then there exists the $\Gamma\text{-}\lim_{\varepsilon\to 0+} G^j_\varepsilon(u,v) = G^j(u,v)$ with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence, where

$$G^j(u,v) = \begin{cases} \displaystyle\int_\Omega \varphi_j(|\nabla u|)dx + \int_{S_u} \vartheta_j(|u^+ - u^-|)d\mathcal{H}^{n-1} + j|D^c u|(\Omega) \\ \qquad\qquad \text{if } u \in GBV(\Omega) \text{ and } v = 1 \text{ a.e.} \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

Since the functionals G^j converge increasingly to G, they also Γ-converge to G as $j \to +\infty$. By the metrizable character of Γ-convergence, we can then find a sequence (ε_j) of real numbers converging to 0 such that $G^j_{\varepsilon_j}$ Γ-converges to G, that is, the thesis. □

Remark 5.4 If φ is convex and even, ϑ is concave and even, and

$$\lim_{t\to+infty} \frac{\varphi(t)}{t} = \lim_{t\to o^+} \frac{\vartheta(t)}{t} = M,$$

then there exist (φ_j), (ψ_j) (k_j) and (ε_j) such that the functionals G_j defined above Γ-converge with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence to

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega \varphi(|\nabla u|)dx + \int_{S_u} \vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + M|D^c u|(\Omega) \\ \qquad\qquad \text{if } u \in GBV(\Omega) \text{ and } v = 1 \text{ a.e.} \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

The proof can be obtained directly from Remark 5.2, using the approximation argument of Proposition 5.3.

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