An H^2 type Riemannian metric on the space of planar curves *

Jayant Shah
Mathematics Department, Northeastern University, Boston MA
email: shah@neu.edu

Abstract

An H^2 type metric on the space of planar curves is proposed and equation of the geodesic is derived. A numerical example is given to illustrate the differneces between H^1 and H^2 metrics.

1 Introduction

Riemannian geometry has by now become a well-established tool for studying variations in shapes. A general differential geometric framework for studying deformations of plane curves has been set up by Mumford and Michor in [3,4] and by Younes et al in [7,8]. The simplest of Riemannian metrics, namely the H^0 metrics, have been studied in [3,10,13,14]. H^1 metrics have been proposed in [2,9,11] and recently, Michor el al have derived a formula for computing the geodesic distances between curves in this metric [6]. The purpose of the present paper is to extend the H^2 metric proposed in [2] for inelastic curves to elastic curves and derive the equation of its geodesic (§3). It is compared with the H^1 metric by means of a numerical example in §4. A scale-invariant version of the H^1 metric given in [9] is described in §2. Extension of the techniques described in this paper to 3D shapes is nontrivial and beyond the scope of this paper; however, see Michor and Mumford [5] for a general framework.

2 H^1 metric

2.1 Definition

Let S denote the unit circle parametrized by the polar angle ω. Let $c(\omega) : S \to \mathbb{R}^2$ be a smooth immersion of degree 1. For a function f on S, let f' denote its derivative $du/d\omega$. Identify $\mathbb{R}^2$ with the complex plane so that $c(\omega) = x(\omega) + iy(\omega)by$. Then, c' may be written as $e^{z(\omega)}$ where $z(\omega) = \lambda(\omega) + i\theta(\omega)$.

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We will denote z also by the notation (λ, θ). Since θ is determined only modulo 2π, we normalize it a proper choice of coordinates in $\mathbb{R}^2$ such that

$$\frac{1}{2\pi}\int_S \theta e^{\lambda} d\omega = \pi \tag{1}$$

The infinitesimal arclength $ds = e^{\lambda} d\omega$. Since the image of c is a closed curve, z satisfies the closure condition

$$\int_S e^z d\omega = \int_S c' d\omega = c(2\pi) - c(0) = 0 \tag{2}$$

Conversely, given $z(\omega)$ satisfying conditions (1) and (2), the immersion may be recovered modulo translation in $\mathbb{R}^2$ by setting

$$x(\omega) + iy(\omega) = \int_0^{\omega} e^{z(\varpi)} d\varpi \tag{3}$$

Let Θ be the set of complex smooth functions of ω such that if $\lambda(\omega)+i\theta(\omega)\epsilon\Theta$, then, λ is 2π-periodic and $\theta(\omega + 2\pi) = \theta(\omega)$. Let

$$\Gamma = (\Gamma_0, \Gamma_1, \Gamma_2) : \Theta \rightarrow \mathbb{R}^3 \tag{4}$$

be defined by setting

$$\Gamma_0 = \frac{1}{2\pi}\int_S \theta e^{\lambda} d\omega - \pi, \qquad \Gamma_1 = \int \cos\theta e^{\lambda} d\omega, \qquad \Gamma_2 = \int \sin\theta e^{\lambda} d\omega \tag{5}$$

Then $\mathbb{E} = \Gamma^{-1}(0)$ is the space of smooth immersions and the orbit space $\mathbb{B}$ of $\mathbb{E}$ under the action of diffeomorphisms of S is the space of "smooth" planar curves with crossings, modulo translations.

The tangent space $T\Theta_z$ at a point $z\epsilon\Theta$ is isomorphic to Θ itself. A tangent vector at z is just a pair of functions μ and φ on S. The function μ specifies the stretching of the curve while φ specifies the rotation of the tangent vectors of the curve. Given tangent vectors $u_1 = (\mu_1 + i\phi_1)$ and $u_2 = (\mu_2 + i\phi_2)\epsilon T\Theta_z$, define a Riemannian metric $<< .,. >>$ on Θ and, by restriction also on $\mathbb{E}$, by setting

$$< u_1, u_2 >= \frac{1}{\ell}\int_S (A^2\mu_1\mu_2 + \varphi_1\varphi_2)e^{\lambda} d\omega \quad \text{where} \quad \ell = \int_S e^{\lambda} d\omega \tag{6}$$

Here A is a constant. The energy of infinitesimal deformation of z by a vector u is given by $\frac{1}{2} << u, u >>$. The larger the value of A, the larger the penalty for stretching. This metric is the same as that defined by Younes and Trouvé in [11] where A is equal to 1. The metric is invariant under the action of $Diff(S)$ and hence defines a metric on $\mathbb{B}$.

2.2 Closure of open curves in H^1 metric

A generic $z\epsilon\Theta$ defines an open curve by means of Equation (3). It may be closed by minimizing $\frac{1}{2}|\Gamma|_\alpha^2 = \alpha\Gamma_0^2 + \Gamma_1^2 + \Gamma_2^2$ where α is a weight. The equation of gradient descent is

$$\left(\frac{\partial\lambda}{\partial t}, \frac{\partial\mu}{\partial t}\right) = \beta\left(\alpha\Gamma_0\nabla\Gamma_0 + \Gamma_1\nabla\Gamma_1 + \Gamma_2\nabla\Gamma_2\right) \tag{7}$$

where

$$\nabla\Gamma_0 = \frac{\ell}{2\pi}\left(\frac{\theta}{A^2}, 1\right) \quad \nabla\Gamma_1 = \ell\left(\frac{\cos\theta}{A^2}, -\sin\theta\right) \quad \nabla\Gamma_2 = \ell\left(\frac{\sin\theta}{A^2}, \cos\theta\right) \tag{8}$$

The vectors $\nabla\Gamma_0, \nabla\Gamma_1, \nabla\Gamma_2$ span the space normal to the level set $\Gamma^{-1}(\Gamma(z))$.

2.3 Geodesic equations in Θ and $\mathbb{E}$

Let $t \to z(t) = (\lambda(t), \theta(t))$ be a path in Θ. Let subscript t denote the partial derivative with respect to t. If f is function on S, let $\overline{f}$ denote its average at a point $z\epsilon\Theta$ defined as $\frac{1}{\ell}\int_S fe^{\lambda}d\omega$. Then the path is a geodesic in Θ if and only if satisfies the equations

$$\begin{aligned} \kappa_\lambda &= \ell(\lambda_t/\ell)_t + \frac{1}{2}\left(\lambda_t^2 + \overline{\lambda_t^2}\right) - \frac{1}{2A}\left(\theta_t^2 - \overline{\theta_t^2}\right) = 0 \\ \kappa_\theta &= (\theta_t/\ell)_t + \lambda_t\theta_t = 0 \end{aligned} \tag{9}$$

which are scale-invariant version of the equations given in [9]. If the path is in $\mathbb{E}$, then it is a geodesic if and only if the projection $P_{T\mathbb{E}}(\kappa)$ of κ onto the tangent space $T\mathbb{E}$ is zero. Let J be the matrix whose $(i,j)^{th}$ entry is $<< \nabla\Gamma_i, \nabla\Gamma_j >>$, $i,j = 0,1,2$. Let $\nabla\Gamma$ denote the matrix whose rows are $\nabla\Gamma_0$, $\nabla\Gamma_1$ and $\nabla\Gamma_2$. Then,

$$P_{T\mathbb{E}}(\kappa) = \kappa - (\nabla\Gamma)^T J^{-1}\begin{bmatrix} << \nabla\Gamma_0, \kappa >> \\ << \nabla\Gamma_1, \kappa >> \\ << \nabla\Gamma_2, \kappa >> \end{bmatrix} \tag{10}$$

(See [9] for details.) If the path $t \to z(t) = (\lambda(t), \theta(t))$ in $\mathbb{E}$ is not a geodesic, we can apply gradient descent to it to find a geodesic path:

$$\frac{\partial z}{\partial\tau} = \beta P_{T\mathbb{E}}(\kappa) \tag{11}$$

The constant β should be sufficiently small to ensure numerical stability of the equation.

3 H^2 metric

A straightforward way to define an H^2 metric on Θ is to introduce derivatives in the metric and set

$$<< u_1, u_2 >>= \ell\int_S (A^2\mu_1'\mu_2' + \varphi_1'\varphi_2')e^{-\lambda}d\omega \tag{12}$$

However, a simpler alternative is to replace the parameter θ by curvature [9]. An embedding $c(\omega) : S \to \mathbb{R}^2$ defines a function $k(\omega)$ on S where $k(\omega)$ is the curvature of the curve at $c(\omega)$. Since $k = d\theta/ds = e^{-\lambda} d\theta/d\omega$, the curvature function $k(\omega)$ satisfies the relation

$$\int_S k e^{\lambda} d\omega = 2\pi \tag{13}$$

The immersion $c(\omega)$ may be recovered modulo rotation and translation by setting $\theta(\omega) = \int_0^{\omega} k e^{\lambda} d\varpi$ and using Equation (3).

Let Ξ now be the set of vector valued functions $\varsigma = (\lambda, k)$ on S. Define

$$\Gamma = (\Gamma_0, \Gamma_1, \Gamma_2) : \Xi \to \mathbb{R}^3 \tag{14}$$

by setting

$$\Gamma_0 \;=\; \int_S k e^{\lambda} d\omega - 2\pi,\ \Gamma_1 = \int \cos\theta e^{\lambda} d\omega,\ \Gamma_2 = \int \sin\theta e^{\lambda} d\omega \tag{15}$$

Then $\mathbb{F} = \Gamma^{-1}(0)$ is the space of smooth immersions modulo translations and rotations in $\mathbb{R}^2$. The orbit space $\mathbb{D}$ of $\mathbb{F}$ under the action of diffeomorphisms of S is now the space of smooth planar curves with crossings, modulo translations and rotations.

The tangent space $T\Xi_\varsigma$ at a point $\varsigma \epsilon \Xi$ is again isomorphic to Ξ itself. Given tangent vectors $u_1 = (\mu_1 + ih_1)$ and $u_2 = (\mu_2 + ih_2)\ \epsilon T\Xi_\varsigma$, define a Riemannian metric $<< .,. >>$ on Ξ and, by restriction also on $\mathbb{F}$, by setting

$$< u_1, u_2 >= \int_S (\frac{A^2}{\ell} \mu_1 \mu_2 + \ell h_1 h_2) e^{\lambda} d\omega \text{ where } \ell = \int_S e^{\lambda} d\omega \tag{16}$$

The energy of deformation now the stretching and bending energy of an elastica. The metric physically makes more sense than the H^1 metric which attaches cost to absolute rotation of the tangent vectors rather than their rotation relative to their immediate neighbors. The new metric is invariant under the action of $Diff(S)$ and hence defines a metric on $\mathbb{D}$.

3.1 Closure of open curves in H^2 metric

As before, an open curve corresponding to a generic $\varsigma \epsilon \Xi$ may be closed by minimizing $\frac{1}{2}|\Gamma|^2_\alpha = \alpha\Gamma_0^2 + \Gamma_1^2 + \Gamma_2^2$. The equation of gradient descent is

$$\left(\frac{\partial \lambda}{\partial t}, \frac{\partial h}{\partial t}\right) = \beta\left(\alpha \Gamma_0 \nabla \Gamma_0 + \Gamma_1 \nabla \Gamma_1 + \Gamma_2 \nabla \Gamma_2\right) \tag{17}$$

where

$$\nabla\Gamma_0 = \left(\frac{\ell k}{A^2}, \frac{1}{\ell}\right) \qquad \nabla\Gamma_1 = \left(\frac{\ell}{A^2}\{\cos\theta - k\gamma_2\}, -\frac{\gamma_2}{\ell}\right)$$

$$\nabla\Gamma_2 \;=\; \left(\frac{\ell}{A^2}\{\sin\theta + k\gamma_1\}, \frac{\gamma_1}{\ell}\right) \tag{18}$$

$$\gamma_i(\omega) = x_i(2\pi) - x_i(\omega), \qquad i = 1, 2 \qquad \text{and} \quad x = (x_1, x_2).$$

3.2 Geodesic equations in Ξ and M

A straighforward calculation shows that a path $t \to \varsigma(t) = (\lambda(t), k(t))$ in Ξ is a geodesic if and only if it satisfies the equations

$$\begin{aligned} \kappa_\lambda &= \ell(\lambda_t/\ell)_t + \frac{1}{2}\left(\lambda_t^2 + \overline{\lambda_t^2}\right) - \frac{\ell^2}{2A^2}\left(k_t^2 + \overline{k_t^2}\right) = 0 \\ \kappa_k &= \frac{1}{\ell}(\ell k_t)_t + \lambda_t k_t = 0 \end{aligned} \tag{19}$$

A path in $\mathbb{F}$ is a geodesic if and only if the projection $P_{T\mathbb{F}}(\kappa) = 0$. The matrix J and the projection $P_{T\mathbb{F}}(\kappa)$ are given by the same formulae as in the case of H^1 metric. If the path $t \to \varsigma(t) = (\lambda(t), k(t))$ in $\mathbb{F}$ is not a geodesic, apply gradient descent to it to find a geodesic path:

$$\frac{\partial \varsigma}{\partial \tau} = \beta P_{T\mathbb{F}}(\kappa) \tag{20}$$

4 A numerical Example

The distance between two unparametrized curves may be found in two steps:

Step 1: Pick parametrizations for the two curves and find the geodesic distance between the two by constructing a minimal geodesic path between the two. Picking the two parametrizations essentially amounts to fixing a point-to-point correspondence between the two curves.

Step 2: Minimize the distance in Step 1 over the set of parametrizations of the second curve, keeping the parametrization of the first curve fixed. Note that if correspondence between a set of landmarks on the two curves is given a priori, it may be incorporated as a constraint on permissible reparametrizations.

The numerical example below finds a geodesic between two parametrized curves with no guarantee that the geodesic found in this way is minimal. One may employ dynamic programming to carry out the second step as described in [11].

There is also the question of choosing the value of the parameter A. It is easy to see that the metrics become degenerate if $A = 0$. For example, in the case of the H^2 metric, if $A = 0$, we can pick $\lambda(t)$ which, in the limit, blows up a point on S where the curvature equals the average curvature and shrinks the rest to a point, thus deforming the given curve into a circle with path length equal to zero. In the same way, in the case of the H^1 metric with $A = 0$, we can deform the curve into an equilateral triangle at no cost. Therefore, we should expect numerical difficulties if the value of A is too small. In fact, for small values of A, the length of the path continued to decrease during gradient descent while the value of κ blew up.

The results of our numerical experiment are shown in the following figure where each column shows a geodesic path constructed from bottom to top. After each step of gradient descent, the curve-closing algorithm was applied to prevent a slow drift away from the manifold of closed curves. The parameter space S was divided into 250 equal parts and a path between the two curves was constructed by interpolating 49 curves between the initial curve and final curve. In the figure, the first two columns show the numerical geodesics in the H^1 metric with values of A equal to 1 (first column) and 10 (second column). The last two columns show the numerical geodesics in the H^2 metric using $A = 40$ (third column) and $A = 100$ (last column). In both cases, some protrusions of the fish are fashioned into horse's legs, head and the tail while the rest of the protrusions are smoothed out. In the case of the H^2 metric, there is a noticeable tendency for the curve to become circular, especially for the lower value of A. In the case of H^1 metric, the curve tends to remain polygonal.

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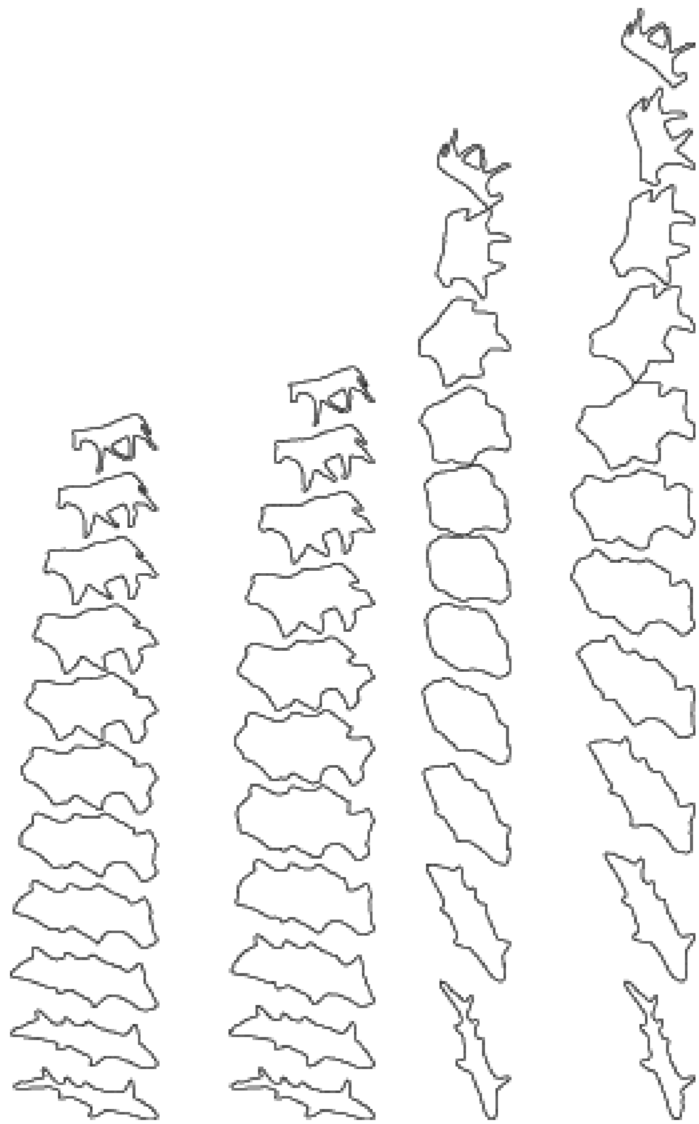

Figure 1: Geodesic paths. H^1, A=1,10, H^2, A=40,200.