AN H^2 RIEMANNIAN METRIC ON THE SPACE OF PLANAR CURVES MODULO SIMILITUDES

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Abstract. Analyzing shape manifolds as Riemannian manifolds has been shown to be an effective technique for understanding their geometry. Riemannian metrics of the type H^0 and H^1 on the space of planar curves have already been investigated in detail. Since in many applications, the basic shape of an object is understood to be independent of its scale, orientation or placement, we consider here an H^2 metric on the space of planar curves modulo similitudes. The metric depends purely on the bending and stretching of the curve. Equations of the geodesic for parametrized curves as well as un-parametrized curves and bounds for the sectional curvature are derived. Equations of gradient descent are given for constructing the geodesics between two given curves numerically.

KEYWORDS: Infinite dimensional manifolds, planar loops, geodesics, sectional curvature

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1. Introduction

One of the approaches to shape analysis is to study the Riemannian geometry of the shape manifold. Examples of this approach may be seen in [2-6, 8-16]. In particular, there has been a great deal of progress in understanding the geometry of planar shapes exemplified by silhouettes and MRI. Mathematically, this space is the space of smooth planar curves. Michor and Mumford [3,4] have analyzed this space in considerable detail. In particular, they derive the geodesic equation for a general Sobolev metric. A more detailed analysis including the sectional curvature of specific metrics is carried out in [9,16]. Conformal versions of the H^0-metric of Michor and Mumford are studied in [9] while an H^1-metric is studied in [16].

It is useful to separate out the pose and scale of an object from its inherent geometry. This leads us to study the space of planar curves modulo translation, rotation and scale. An example of such a formulation is an H^1-metric analyzed in [16] by Younes et al. To obtain a metric which depends purely on the bending and stretching of the curve, what is needed is an H^2-metric. Klassen et al studied such a metric in [2]. This was further investigated by the author in [8]. The current paper is a more thorough analysis of a closely related H^2-metric in the framework of Michor and Mumford. We derive the geodesic equation and give the equation of gradient descent for deforming a given curve into a geodesic. Using the techniques developed in [3,4,16], we compute the sectional curvature and derive an absolute bound which guarantees the existence of minimal geodesics.

This paper is organized as follows. In §2, we define the space of parametrized closed curves and describe the action of the group of similitudes. We define Sobolev bilinear forms on the tangent bundle which induces Riemannian metrics on the quotient space. In the rest of the paper, we restrict the analysis to the case of H^2 metric.

In §3, we derive the geodesic equation for the space of parametrized curves modulo similitudes.

In §4, the action of the reparametrization group is considered and the equation for the geodesics on the quotient by the group is derived.

Computation of the sectional curvature is rather involved and is carried out in several steps. In §5.1, local charts are constructed by means of a much bigger space $\widetilde{\Omega}$ which includes open curves. Rotations of the curves induce translations in $\widetilde{\Omega}$. The space Ω is the quotient of $\widetilde{\Omega}$ by these translations. The space of closed curves modulo similitudes appears as a submanifold Ω_0 of Ω. In §5.2, we define the tangent bundles on $\widetilde{\Omega}$ and Ω. We identify a subbundle of the tangent space of $\widetilde{\Omega}$ with the tangent bundle of Ω, allowing us to carry out all our computations on this subbundle. We carry over the Sobolev bilinear form to $\widetilde{\Omega}$ which then defines a Sobolev metric on Ω. In §5.3, we describe the action of the group of reparametrization on $\widetilde{\Omega}$ and Ω. The metric on Ω induces a metric on its quotient by the group. Christoffel Symbols are computed in §5.4. The sectional curvature of Ω is computed in §5.5 and an application of Gauss Lemma in §5.6 gives a formula for the sectional curvature of the submanifold Ω_0. In §5.7, we use O'Neill's formula to compute the sectional curvature of Ω_0 modulo the group of reparametrizations. In §5.7, we derive an absolute bound on the sectional curvature.

2. Sobolev Metrics

Let $\mathrm{Imm}(S^1, \mathbb{R}^2)$ denote the space of all C^∞ immersions $c : S^1 \to \mathbb{R}^2$.

Let the unit circle S^1 be parametrized by θ. Let s denote the arc-length along the image curve of c in $\mathbb{R}^2$. The infinitesimal arc-length $ds = |c_\theta|\, d\theta$. (Subscripts θ, s, t denote the derivative.)

A tangent vector h at a point $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ is just a vector field in $\mathbb{R}^2$ along c in $\mathbb{R}^2$. If a, b are vectors in $\mathbb{R}^2$, $a \cdot b$ will denote their dot product. Let v and n denote the vector fields of unit tangent and unit normal vectors along c. As vector fields along c viewed as a curve in the complex plane $\mathbb{C}$, $v = c_s$ and $n = ic_s$. We will often represent h_s as a complex function:

$$h_s = (h_s \cdot v)v + (h_s \cdot n)n = ((h_s \cdot v) + i(h_s \cdot n))\, c_s$$

The vector $(h_s \cdot v, h_s \cdot n)$ as a complex function takes the form

$$h_s \cdot v + ih_s \cdot n = h_s / c_s$$

We will use the following notations interchangeably:

$$(m_s \cdot v, m_s \cdot n) \cdot (h_s \cdot v, h_s \cdot n) = (m_s \cdot v)(h_s \cdot v) + (m_s \cdot n)(h_s \cdot n) = \frac{m_s}{c_s} \cdot \frac{h_s}{c_s}$$

Let Σ denote the group of similitudes. Its action on c as a curve in $\mathbb{C}$ is given by $\alpha c + \beta$ where α, β are (complex) constants. The vertical vectors at c of the quotient map

$$\rho : \mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$$

have the form $\alpha c+\beta$. If h is a vertical vector, then h_s/c_s is constant, that is, $h_s\cdot v$ and $h_s\cdot n$ are constant. We define Σ-horizontal vectors at c as

$$\{h|\overline{h}=\overline{h_s\cdot v}=\overline{h_s\cdot n}=0\} \tag{2.1}$$

where the bar over a variable denotes its average value:

$$\overline{f}=\frac{1}{\ell}\int_c f ds,\qquad \ell=\int_c ds$$

Let

$$f^0=f-\overline{f}$$

The horizontal and the vertical vectors define a decomposition of the tangent bundle of $\mathrm{Imm}(S^1,\mathbb{R}^2)$ into two subbundles such that the tangent map

$$T_c\,\mathrm{Imm}(S^1,\mathbb{R}^2)\to T_{\rho(c)}\,\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$$

is an isomorphism when restricted to the horizontal vectors. We identify the tangent vectors at $\rho(c)$ with the horizontal tangent vectors at c. The projection from the tangent space at c onto the space of horizontal tangent vectors is given by the formula

$$h\longmapsto h^\Sigma=h^0-(\overline{h_s/c_s})c^0 \tag{2.2}$$

Note that c^0 is the curve c translated so that its center of gravity is at the origin.

Given a path $c(t)$ in $\mathrm{Imm}(S^1,\mathbb{R}^2)$, there exists a horizontal path $c^\Sigma(t)=\alpha(t)c(t)+\beta(t)$ where $\alpha(t)$ and $\beta(t)$ are given by equations

$$\begin{aligned}\alpha_t+\alpha\overline{c_{ts}/c_s} &= 0\\ \beta_t+\overline{(\alpha c)_t} &= 0\end{aligned}$$

such that $c(t)$ and $c^\Sigma(t)$ project to the same path in $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$.

We now define a bilinear form on the tangent space at c:

$$\langle m,h\rangle=\ell^{2p-3}\int_c(D_s^{p-1}(m_s^\Sigma\cdot v),D_s^{p-1}(m_s^\Sigma\cdot n))\cdot(D_s^{p-1}(h_s^\Sigma\cdot v),D_s^{p-1}(h_s^\Sigma\cdot n))ds$$

where $D_s=d/ds$, ℓ is the length of c and $p\geq 1$. Note that $h_s^\Sigma\cdot v=(h_s\cdot v)^0$ and $h_s^\Sigma\cdot n=(h_s\cdot n)^0$. Moreover $\langle h,h\rangle=0$ if and only if h is a vertical vector. $\langle m,h\rangle$ is non-degenerate on the horizontal subbundle and defines a Riemannian metric on it and hence on $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$.

The case $p=1$ has been treated in [16] using a representation of $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$ by grassmannians. If $p\geq 2$, $D_s^{p-1}(h_s^\Sigma\cdot v,h_s^\Sigma\cdot n)=D_s^{p-1}(h_s\cdot v,h_s\cdot n)$. We may assume $\ell=1$ since the metric is scale-invariant. Setting $p=2$ from now on, we consider the metric induced by the bilinear form

$$\langle m,h\rangle=\int_c(m_s\cdot v,m_s\cdot n)_s\cdot(h_s\cdot v,h_s\cdot n)_s ds \tag{2.3}$$

Given a path $c(t):[0,1]\to\mathrm{Imm}(S^1,\mathbb{R}^2)$, we define its length by setting

$$L(c(t))=\int_0^1\sqrt{\langle c_t,c_t\rangle}dt$$

The length of a path is invariant under the action of Σ.

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3. Geodesic Equation for the H^2-metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$

Let $c(t) : [0,1] \to \mathrm{Imm}(S^1, \mathbb{R}^2)$ be a horizontal lift of a path in $\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$. We may assume that for each t, the curve $c(t)$ has unit length. Deformation of a curve by a horizontal tangent vector preserves its length. We derive the geodesic equation by calculating the first variation of the energy

$$\begin{aligned} E(c) &= \frac{1}{2}\int_0^1 \langle c_t, c_t \rangle \, dt \\ &= \frac{1}{2}\int_0^1 \int_c ((c_{ts}\cdot v)_s)^2 + ((c_{ts}\cdot n)_s)^2 \, dsdt \end{aligned}$$

where $c_{ts} = (c_t)_s = D_s D_t c$.

Let $m(t)$ be a field of horizontal tangent vectors along $c(t)$, vanishing at its end-points. We have the following formulas for derivatives in the direction of m:

$$\begin{aligned} D_m|c_\theta| &= (m_s\cdot v)|c_\theta|, \quad D_m D_s = -(m_s\cdot v)D_s + D_s D_m \\ D_m v &= (m_s\cdot n)n, \quad D_m n = -(m_s\cdot n)v \\ D_m(c_{ts}\cdot v) &= D_m D_t \log|c_\theta| = D_t D_m \log|c_\theta| = (m_s\cdot v)_t, \quad D_m(c_{ts}\cdot n) = (m_s\cdot n)_t \end{aligned}$$

$$\begin{aligned} D_m E(c) &= \frac{1}{2}\int_0^1 D_m \int_{S^1} \left((c_{ts}\cdot v)_\theta^2 + (c_{ts}\cdot n)_\theta^2\right) \frac{d\theta}{|c_\theta|} dt \\ &= \int_0^1 \int_{S^1} ((m_s\cdot v)_{\theta t}(c_{ts}\cdot v)_\theta + (m_s\cdot n)_{\theta t}(c_{ts}\cdot n)_\theta) \frac{d\theta}{|c_\theta|} dt \\ &\quad -\frac{1}{2}\int_0^1 \int_{S^1} (m_s\cdot v)\left((c_{ts}\cdot v)_\theta^2 + (c_{ts}\cdot n)_\theta^2\right) \frac{d\theta}{|c_\theta|} dt \\ &= -\int_0^1 \int_c ((m_s\cdot v)_s(c_{ts}\cdot v)_{st} + (m_s\cdot n)_s(c_{ts}\cdot n)_{st}) \, dsdt \\ &\quad -\frac{1}{2}\int_0^1 \int_c (m_s\cdot v)\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right) dsdt \end{aligned}$$

where the last step follows from integration by parts. Since m is horizontal, m_s/c_s has zero mean and hence,

$$\begin{aligned} D_m E(c) &= -\int_0^1 \int_c \left((m_s\cdot v)_s((c_{ts}\cdot v)_{st})^0 + (m_s\cdot n)_s((c_{ts}\cdot n)_{st})^0\right) dsdt \\ &\quad -\frac{1}{2}\int_0^1 \int_c (m_s\cdot v)\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right)^0 dsdt \end{aligned}$$

The operator D_s^{-1} is uniquely defined on zero-mean functions by requiring that $\overline{D_s^{-1} f} = 0$. Applying integration by parts,

$$\begin{aligned} D_m E(c) &= -\int_0^1 \int_c \left((m_s\cdot v)_s((c_{ts}\cdot v)_{st})^0 + (m_s\cdot n)_s((c_{ts}\cdot n)_{st})^0\right) dsdt \\ &\quad +\frac{1}{2}\int_0^1 \int_c (m_s\cdot v)_s D_s^{-1}\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right)^0 dsdt \\ &= -\int_0^1 \langle m, \gamma(c)\rangle \, dt \end{aligned}$$

where the symbol γ denotes the geodesic curvature of the path in $\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$ corresponding to the path $c(t)$. An explicit expression for γ may be derived as follows.

Using the identity $f_{st} = -(c_{ts} \cdot v) f_s + f_{ts}$, we get

$$\begin{aligned}
(c_{ts} \cdot v)_{st} &= (c_{ts} \cdot v)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\
&= \left(c_{tts} \cdot v - (c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2\right)_s - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\
&= \left(c_{tts} \cdot v - \frac{3}{2}(c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2\right)_s = ((c_{ts} \cdot v)_{st})^0
\end{aligned}$$

$$\begin{aligned}
(c_{ts} \cdot n)_{st} &= (c_{ts} \cdot n)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot n)_s \\
&= (c_{tts} \cdot n - 2(c_{ts} \cdot v)(c_{ts} \cdot n))_s - (c_{ts} \cdot v)(c_{ts} \cdot n)_s
\end{aligned}$$

$$((c_{ts} \cdot n)_{st})^0 = \left(c_{tts} \cdot n - 2(c_{ts} \cdot v)(c_{ts} \cdot n) - D_s^{-1}\left((c_{ts} \cdot v)(c_{ts} \cdot n)_s\right)^0\right)_s$$

Let Λ be the vector (Λ_1, Λ_2) where

$$\begin{aligned}
\Lambda_1 &= \left(\frac{3}{2}(c_{ts} \cdot v)^2 - (c_{ts} \cdot n)^2\right) + \frac{1}{2} D_s^{-2}\left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2\right)^0 \\
\Lambda_2 &= (2(c_{ts} \cdot v)(c_{ts} \cdot n)) + D_s^{-1}\left((c_{ts} \cdot v)(c_{ts} \cdot n)_s\right)^0
\end{aligned}$$

Then,

$$D_m E(c) = -\int_0^1 \int_c \left(\frac{m_s}{c_s}\right)_s \cdot \left(\frac{c_{tts}}{c_s} - \Lambda\right)_s ds$$

Since $\int_c m_s ds = 0$, we have

$$0 = \int_c (m_s/c_s)^* c_s^* ds = \int_c (m_s/c_s)_s^* D_s^{-1}(c_s^*) ds$$

where the superscript $*$ denotes the complex conjugate. Therefore,

$$\int_c \left(\frac{m_s}{c_s}\right)_s \cdot D_s^{-1}(c_s^*) ds = 0$$

and

$$\begin{aligned}
D_m E(c) &= -\int_0^1 \int_c \left(\frac{m_s}{c_s}\right)_s \cdot \left(\frac{c_{tts}}{c_s} - \Lambda - \alpha D_s^{-2}(c_s^*)\right)_s ds \\
&= -\int_0^1 \int_c \left(\frac{m_s}{c_s}\right)_s \cdot \left(\frac{\gamma_s}{c_s}\right)_s ds
\end{aligned}$$

for all $\alpha \in \mathbb{C}$. Let $\gamma_s/c_s = c_{tts}/c_s - \Lambda - \alpha D_s^{-2}(c_s^*)$. Since we must have $\int_c \gamma_s ds = 0$, set

$$\alpha = \frac{\int_c \Lambda c_s ds}{\int_c \left|D_s^{-1}(c_s)\right|^2 ds}$$

Finally,

$$\gamma = c_{tt} - D_s^{-1}\left(\left(\Lambda + \alpha D_s^{-2}(c_s^*)\right) c_s\right) \tag{3.1}$$

A path $c(t)$ is a geodesic if and only if

$$c_{tt} = D_s^{-1}\left(\left(\Lambda + \alpha D_s^{-2}(c_s^*)\right) c_s\right) \tag{3.2}$$

A path $c(t)$ may be deformed into a geodesic by gradient descent: $\frac{\partial c}{\partial \tau} \propto \gamma$.

An alternative form of the geodesic equation and the gradient descent without using the explicit form of γ may be derived as follows:

$$\text{(3.3)} \qquad \text{Let} \quad \mu = (c_{ts} \cdot v)_{st} - \frac{1}{2} D_s^{-1} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0, \quad \nu = (c_{ts} \cdot n)_{st}$$

Then,

$$\left(\frac{c_{tts}}{c_s} - \Lambda \right)_s = (\mu, \nu^0)$$

$$\text{(3.4)} \qquad \begin{aligned} \langle m, \gamma \rangle &= \int_c \left(\frac{m_s}{c_s} \right)_s \cdot \left(\frac{c_{tts}}{c_s} - \Lambda \right)_s ds \\ &= \int_c \left((m_s \cdot v)_s \, \mu + (m_s \cdot n)_s \, \nu^0 \right) ds \\ &= - \int_c \left((m_s \cdot v) \, \mu_s + (m_s \cdot n) \, \nu_s \right) ds \\ &= - \int_c \left((m_s \cdot v) \, v + (m_s \cdot n) \, n \right) \cdot (\mu_s v + \nu_s n) \, ds \\ &= - \int_c m_s \cdot (\mu_s v + \nu_s n) \, ds \\ &= \int_c m \cdot (\mu_s v + \nu_s n)_s \, ds \end{aligned}$$

for all vectors $m \in T_c \operatorname{Imm}(S^1, \mathbb{R}^2)$.

$$\text{(3.5)} \qquad \begin{aligned} & (\mu_s v + \nu_s n)_s \\ = & \left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s \\ = & \left((c_{ts}/c_s)_{sts} c_s \right)_s - \frac{1}{2} \left((\|(c_{ts}/c_s)_s\|^2)^0 c_s \right)_s \end{aligned}$$

Since

$$\gamma(c) = 0 \Longleftrightarrow \langle m, \gamma(c) \rangle = 0 \quad \text{for all } m \in T_c \operatorname{Imm}(S^1, \mathbb{R}^2),$$

a path $c(t)$ is a geodesic if and only if

$$\text{(3.6)} \qquad \left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s = 0$$

Since

$$\begin{aligned} \left\langle \left((\mu_s v + \nu_s n)_s \right)^\Sigma, \gamma \right\rangle &= \langle (\mu_s v + \nu_s n)_s, \gamma \rangle \geq 0 \\ \text{and} \qquad D_{m^\Sigma} E(c) &= - \int_0^1 \left\langle m^\Sigma, \gamma \right\rangle dt = - \int_0^1 \langle m, \gamma \rangle \, dt \end{aligned}$$

the gradient descent equation takes form

$$\text{(3.7)} \qquad \frac{\partial c}{\partial \tau} \propto \left(\left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s \right)^\Sigma$$

The geodesic equation may also be written in terms of the "momentum" vector u as follows:

$$u = \left((c_{ts} \cdot v)_{ss} \, v + (c_{ts} \cdot n)_{ss} \, n \right)_s = \left((c_{ts}/c_s)_{ss} c_s \right)_s$$

$$\begin{aligned} u_t &= ((c_{ts}/c_s)_{ss}c_s)_{st} \\ &= -(c_{ts}\cdot v)\,u + ((c_{ts}/c_s)_{ss}c_s)_{ts} \\ &= -(c_{ts}\cdot v)\,u + ((c_{ts}/c_s)_{sst}c_s)_s + i\,((c_{ts}\cdot n)\,(c_{ts}/c_s)_{ss}c_s)_s \end{aligned}$$

where we have used the formula $c_{st} = D_t(c_s) = i(c_{ts}\cdot n)c_s$. Therefore,

$$\begin{aligned} u_t &= -(c_{ts}\cdot v)\,u + (-(c_{ts}\cdot v)\,(c_{ts}/c_s)_{ss}c_s + (c_{ts}/c_s)_{sts}c_s)_s \\ &\quad + i\,((c_{ts}\cdot n)\,u + (c_{ts}\cdot n)_s\,(c_{ts}/c_s)_{ss}c_s) \\ &= (-2c_{ts}\cdot v + ic_{ts}\cdot n)\,u \\ &\quad + (-c_{ts}\cdot v + ic_{ts}\cdot n)_s\,(c_{ts}/c_s)_{ss}c_s + ((c_{ts}/c_s)_{sts}c_s)_s \end{aligned}$$

$$((c_{ts}/c_s)_{sts}c_s)_s = u_t + (2c_{ts}\cdot v - ic_{ts}\cdot n)\,u + (c_{ts}\cdot v - ic_{ts}\cdot n)_s\,(c_{ts}/c_s)_{ss}c_s \tag{3.8}$$

Substituting (3.8) into (3.5) we get

$$\begin{aligned} (\mu_s v + \nu_s n)_s &= u_t + (2c_{ts}\cdot v - ic_{ts}\cdot n)\,u + (c_{ts}\cdot v - ic_{ts}\cdot n)_s\,(c_{ts}/c_s)_{ss}c_s \\ &\quad -\frac{1}{2}\left((\|(c_{ts}/c_s)_s\|^2)^0 c_s\right)_s \end{aligned}$$

Noting $D_s c_s = i\kappa c_s$,

$$\begin{aligned} &(c_{ts}\cdot v - ic_{ts}\cdot n)_s\,(c_{ts}/c_s)_{ss}c_s - \frac{1}{2}\left((\|(c_{ts}/c_s)_s\|^2)^0 c_s\right)_s \\ &= -i[(c_{ts}\cdot v)_{ss}(c_{ts}\cdot n)_s - (c_{ts}\cdot v)_s(c_{ts}\cdot n)_{ss} \\ &\qquad + \frac{\kappa}{2}\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right)^0]c_s \end{aligned}$$

where κ is the curvature of c defined by the equation $v_s = \kappa n$.

$$\begin{aligned} (\mu_s v + \nu_s n)_s &= u_t + (2c_{ts}\cdot v - ic_{ts}\cdot n)\,u \\ &\quad -i[(c_{ts}\cdot v)_{ss}(c_{ts}\cdot n)_s - (c_{ts}\cdot v)_s(c_{ts}\cdot n)_{ss} \\ &\qquad + \frac{\kappa}{2}\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right)^0]c_s \end{aligned} \tag{3.9}$$

The geodesic equation takes the form

$$\begin{aligned} u_t + (2c_{ts}\cdot v - ic_{ts}\cdot n)\,u &= i[(c_{ts}\cdot v)_{ss}(c_{ts}\cdot n)_s - (c_{ts}\cdot v)_s(c_{ts}\cdot n)_{ss} \\ &\qquad + \frac{\kappa}{2}\left((c_{ts}\cdot v)_s^2 + (c_{ts}\cdot n)_s^2\right)^0]c_s \end{aligned} \tag{3.10}$$

4. Geodesic Equation for the H^2-metric on $\mathrm{Imm}(S^1,\mathbb{R}^2)/(\Sigma\times\mathrm{Diff}(S^1))$

The group of diffeomorphisms $\mathrm{Diff}(S^1)$ acts on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ by composition from the right. The action commutes with the action of Σ and hence $\mathrm{Diff}(S^1)$ acts on $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$. The quotient $\mathrm{Imm}(S^1,\mathbb{R}^2)/(\Sigma\times\mathrm{Diff}(S^1))$ is the space of un-parametrized curves modulo similitudes which inherits a Riemannian metric making the quotient map a submersion.

The tangent bundle of $\mathrm{Imm}(S^1,\mathbb{R}^2)/(\Sigma\times\mathrm{Diff}(S^1))$ splits into vertical and horizontal subbundles which are orthogonal with respect to the metric on $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$. The geodesics on $\mathrm{Imm}(S^1,\mathbb{R}^2)/(\Sigma\times\mathrm{Diff}(S^1))$ are just the horizontal geodesics on $\mathrm{Imm}(S^1,\mathbb{R}^2)/\Sigma$. The action of an infinitesimal diffeomorphism corresponds to a vector field bv along c where $b \in \mathbb{R}$. A tangent vector $h \in T_{\rho(c)}\,\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$ is $\mathrm{Diff}(S^1)$-horizontal

$$\begin{aligned}
\iff\quad & \langle bv, h\rangle = \int_c bv \cdot \left((h_s \cdot v)_{ss}\, v + (h_s \cdot n)_{ss}\, n\right)_s ds = \int_c bv \cdot \left((h_s/c_s)_{ss} c_s\right)_s = 0 \\
\iff\quad & v \cdot \left((h_s \cdot v)_{ss}\, v + (h_s \cdot n)_{ss}\, n\right)_s = v \cdot \left((h_s/c_s)_{ss} c_s\right)_s = (h_s \cdot v)_{sss} - \kappa\, (h_s \cdot n)_{ss} = 0
\end{aligned}$$

Suppose $c(t)$ is horizontal with respect to $\Sigma \times \mathrm{Diff}(S^1)$. Then

$$v \cdot \left((c_{ts}/c_s)_{ss} c_s\right)_s = v \cdot u = 0$$

Therefore, the momentum vector $u = an$, $a(t) \in \mathbb{R}$. Substituting this in (3.9), we get

$$\begin{aligned}
(\mu_s v + \nu_s n)_s \quad = \quad & [a_t + 2(c_{ts} \cdot v)a - (c_{ts} \cdot v)_{ss}(c_{ts} \cdot n)_s + (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss} \\
& -\frac{\kappa}{2}\left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2\right)^0]n
\end{aligned}$$

The geodesic equation takes the form

$$\begin{aligned}
a_t + 2(c_{ts} \cdot v)a \quad = \quad & (c_{ts} \cdot v)_{ss}(c_{ts} \cdot n)_s - (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss} \\
& +\frac{\kappa}{2}\left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2\right)^0
\end{aligned} \tag{4.1}$$

Since $\langle bv, \gamma\rangle = \int_c bv \cdot (\mu_s v + \nu_s n)_s\, ds = 0$, the geodesic curvature $\gamma(c)$ is horizontal with respect to $\Sigma \times \mathrm{Diff}(S^1)$.

A horizontal path $c(t)$ may be constructed as follows. Lift $C(t)$ to a path $c(t)$ on $\mathrm{Imm}(S^1, \mathbb{R}^2)$. We now construct a path $\varphi(t, \theta) \in \mathrm{Diff}(S^1)$ such that $c \circ \varphi$ is horizontal. Write $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ and let $\eta = |c_\theta|\, \xi$. Note that $(c \circ \varphi)_t = (c_t + \eta v) \circ \varphi$ and $D_{c\circ\varphi}(f \circ \varphi) = (D_c f) \circ \varphi$ (see [3]). The path $c \circ \varphi$ is horizontal with respect to $\mathrm{Diff}(S^1)$

$$\begin{aligned}
\iff\quad & \left((c_t + \eta v)_s \cdot v\right)_{sss} - \kappa\left((c_t + \eta v)_s \cdot n\right)_{ss} = 0 \\
\iff\quad & L\eta + (c_{ts} \cdot v)_{sss} - \kappa(c_{ts} \cdot n)_{ss} = 0
\end{aligned}$$

where

$$L\eta = \eta_{ssss} - \kappa(\kappa\eta)_{ss} \tag{4.2}$$

The operator L is self-adjoint and it is positive definite provided that c is not a circle. Assuming that the path does not pass through a circle, η may be calculated by minimizing

$$\int_c \frac{1}{2}(\eta_{ss}^2 + (\kappa\eta)_s^2) + ((c_{ts} \cdot v)_{sss} - \kappa(c_{ts} \cdot n)_{ss})\eta ds$$

by gradient descent. Then, integrate the equation $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ to obtain φ and the horizontal path $(c \circ \varphi)^\Sigma$.

5. Sectional Curvature

5.1. **Local Charts.** Let $c : S^1 \to \mathbb{C}$ be an immersion. c_θ is translation invariant.

Σ acts on c_θ by multiplication and on $\log c_\theta$ by translation. Hence, $(\log c_\theta)_\theta$ is Σ-invariant.

$$\int_{S^1} (\log c_\theta)_\theta\, d\theta = \log c_\theta|_{2\pi} - \log c_\theta|_0 = 2\pi J i$$

where J is the rotation index of c. The space of $(\log c_\theta)_\theta$ provides local charts of $\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$ as follows.

Fix an integer J, the rotation index. Let

$$\Omega = \left\{ z \in C^\infty(S^1, \mathbb{R}^2) | \int_{S^1} z_1 d\theta = 0, \int_{S^1} z_2 d\theta = 2\pi J \right\}$$
$$\widetilde{\Omega} = \left\{ z \in C^\infty(\mathbb{R}, \mathbb{R}^2) | z_1, z_2 - Jx \ 2\pi\text{-periodic}, \int_0^{2\pi} e^{z_1} dx = 1 \right\}$$

where z_1, z_2 are the components of z. The map

$$\chi : \widetilde{\Omega} \to \Omega$$

is defined by setting $\chi(z) = z_x$. A point $z \in \widetilde{\Omega}$ defines a curve $c_z : \mathbb{R} \to \mathbb{C}$:

$$c_z(x) = \int_0^x e^{z_1 + iz_2} dx$$

c_z is a closed curve of length 1 if $c_z(2\pi) = 0$. The arclength s of c_z is given by $s(x) = \int_0^x e^{z_1} dx$. The unit tangent vector field v and the unit normal vector field n along c_z are given by e^{iz_2} and ie^{iz_2} respectively. The curvature $\kappa = z_{2s}$.

Let

$$\widetilde{\Omega}_0 = \{z : c_z \text{ is closed}\}$$
$$\Omega_0 = \chi(\widetilde{\Omega}_0)$$

Define a map

$$\Omega_0 \to \mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$$

as follows. Let $\zeta \in \Omega_0$. $\chi^{-1}(\zeta)$ consists of pairs (z_1, z_2) such that z_1 is uniquely determined and z_2 is unique up to a constant. Let $c_{\chi^{-1}(\zeta)} = \left\{ c_z : z \in \chi^{-1}(\zeta) \right\}$. The curves $c_{\chi^{-1}(\zeta)}$ lie in the same $\sum$-orbit and hence map to a unique point in $\mathrm{Imm}(S^1, \mathbb{R}^2)/\Sigma$.

5.2. **Tangent Bundles.** Let $z \in \widetilde{\Omega}$. If a function $f(x)$ is 2π-periodic, we will also regard it as a function on S^1 as well as a function on the circle $\Delta = \{e^{2\pi i s} : 0 \leq s \leq 1\}$. The tangent spaces $T_z\widetilde{\Omega}$ and $T_{\chi(z)}\Omega$ are given by

$$T_z\widetilde{\Omega} = \{h = (h_1, h_2) : h_1, h_2 \ 2\pi\text{-periodic}, \int_\Delta h_1 ds = 0\}$$

$$T_{\chi(z)}\Omega = \{h_\theta : h \in T_z\widetilde{\Omega}\}$$

The vertical vectors at z are of the form $(0, a), a \epsilon \mathbb{R}$. Let

$$T_z^0\widetilde{\Omega} = \{h : \int_\Delta h ds = 0\}$$

We have the decomposition

$$T_z\widetilde{\Omega} \simeq T_z^0\widetilde{\Omega} \oplus \mathbb{R}$$

such that

$$T_{\chi(z)}\Omega \simeq T_z^0\widetilde{\Omega}$$

Every tangent vector field on Ω lifts uniquely to a section of $T^0\widetilde{\Omega}$. We will carry out all the calculations below in terms of the sections of $T^0\widetilde{\Omega}$.

Let

$$m \in T_c \,\mathrm{Imm}(S^1, \mathbb{R}^2)$$

. Then,

$$\begin{aligned} D_m c_\theta &= D_m(|c_\theta|v) \\ &= (m_s \cdot v)|c_\theta|v + (m_s \cdot n)n|c_\theta| \\ &= \{(m_s \cdot v) + i(m_s \cdot n)\}c_\theta \end{aligned}$$

and $D_m \log c_\theta = (m_s \cdot v) + i(m_s \cdot n)$. Therefore, if z_c is the image of c in $\widetilde{\Omega}$, m maps onto $(m_s \cdot v, m_s \cdot n) \in T_{z_c}\widetilde{\Omega}$. The subriemannian metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ induces a subrimannian metric on $\widetilde{\Omega}$. If $m, h \epsilon T_z\widetilde{\Omega}$,

$$\langle m, h \rangle = \int_\Delta m_s \cdot h_s ds$$

The metric is non-degenerate on $T^0\widetilde{\Omega}$ and hence, induces a Riemannian metric on Ω.

5.3. **Action of the Group of Reparametrization.** Let $\mathrm{Diff}^+(\mathbb{R})$ be the group of increasing C^∞ diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x + 2\pi) = \varphi(x) + 2\pi$ for all x. Let $\mathrm{Diff}^0(\mathbb{R})$ be the central group subgroup of translations: $\varphi(x) = x + 2\pi n$. The quotient $\mathrm{Diff}^+(\mathbb{R})/\mathrm{Diff}^0(\mathbb{R})$ is the group of diffeomorphisms of S^1, $\mathrm{Diff}^+(S^1)$. $\mathrm{Diff}^+(\mathbb{R})$ acts on $\widetilde{\Omega}$ as follows:

$$\begin{aligned} z_1^\varphi &= z_1 \circ \varphi + \log \varphi_x \\ z_2^\varphi &= z_2 \circ \varphi \end{aligned}$$

The action commutes with the translations $z_2 \to z_2 + a$ and hence $\mathrm{Diff}^+(\mathbb{R})$ acts on Ω. The action of $\mathrm{Diff}^0(\mathbb{R})$ on Ω is trivial so that $\mathrm{Diff}^+(S^1)$ acts on Ω. The metric on Ω is defined independently of parametrization. Therefore the quotient map

$$\Pi : \Omega \to \Omega/\mathrm{Diff}^+(S^1)$$

is a Riemannian submersion. An infinitesimal diffeomorphism, a, of S^1 induces a tangent vector field bv along c_z where $b = e^{z_1}a$ and maps to a vertical vector $(L_1 b)^0 \in T_z^0\widetilde{\Omega}$ where $L_1 = (D_s, \kappa)$. The condition for $h \in T_z^0\widetilde{\Omega}$ to be Π-horizontal takes the form $L_1^* \cdot h_{ss} = 0$ where the adjoint $L_1^* = (-D_s, \kappa)$. If $h \in T_z^0\widetilde{\Omega}$, let h^V and h^H denote its horizontal and vertical components. h^V has the form $(L_1 b)^0$. Since $L_1^* \cdot (h - L_1 b)_{ss} = 0$,

$$L_1^* \cdot h_{ss} = L_1^* \cdot (L_1 b) = -b_{ssss} + \kappa\,(\kappa b)_{ss} = -Lb$$

where L is the self-adjoint operator defined earlier (Equation 4.2). We have

$$h^V = -\left(L_1 \cdot L^{-1}\left(L_1^* \cdot h_{ss}\right)\right)^0$$

provided that c_z is not a circle.

5.4. **Christoffel Symbols.** Let h, k, m be sections of $T^0\widetilde{\Omega}$ such that $h_\theta, k_\theta, m_\theta$ are constant vector fields on Ω. We calculate the Christoffel symbol $\Gamma(h,k) \in T^0\widetilde{\Omega}$ using the two gradients, $K(m,h)$ and $H(h,k)$ of $D_m \langle h, k \rangle$:

$$D_m \langle h, k \rangle = \langle K(m,h), k \rangle = \langle m, H(h,k) \rangle$$

$$\begin{aligned}\int_\Delta K_s(m,h)\cdot k_s ds &= D_m \int_\Delta h_s\cdot k_s ds\\ &= D_m\int_{S^1} h_\theta\cdot k_\theta e^{-z_1}d\theta\\ &= -\int_{S^1} h_\theta\cdot k_\theta m_1 e^{-z_1}d\theta\\ &= -\int_\Delta h_s\cdot k_s m_1 ds\\ &\quad -\int_\Delta (m_1 h_s)^0\cdot k_s ds\end{aligned}$$

Therefore, $K_s(m,h) = -(m_1h_s)^0$. Similarly,

$$\begin{aligned}\int_\Delta H_s(h,k)\cdot m_s ds &= -\int_\Delta h_s\cdot k_s m_1 ds\\ &= -\int_\Delta (h_s\cdot k_s)^0 m_1 ds\\ &= \int_\Delta m_{1s} D_s^{-1}(h_s\cdot k_s)^0 ds\end{aligned}$$

Hence, $H_s(h,k) = \left(D_s^{-1}(h_s\cdot k_s)^0, 0\right)$. We have the identity

$$\Gamma(h,k) = \frac{1}{2}\left[K(h,k)+K(k,h)-H(h,k)\right]$$

Therefore,

$$\Gamma_s(h,k) = -\frac{1}{2}\left[(h_1k_s)^0 + (k_1h_s)^0 + \left(D_s^{-1}(h_s\cdot k_s)^0, 0\right)\right] \tag{5.1}$$

5.5. **Sectional Curvature of Ω.** Let m, h be sections of $T^0\widetilde{\Omega}$ such that m_θ, h_θ are constant vector fields on Ω. We have the identity for the sectional curvature at the two-dimensional subspace of the tangent space at $\zeta\in\Omega$ spanned by m and h:

$$\begin{aligned}\kappa_{\zeta,,\Omega}(m,h) &= D^2_{m,h}\langle m,h\rangle - \frac{1}{2}D^2_{m,m}\langle h,h\rangle - \frac{1}{2}D^2_{h,h}\langle m,m\rangle\\ &\quad + \langle\Gamma(m,h),\Gamma(m,h)\rangle - \langle\Gamma(m,m),\Gamma(h,h)\rangle\end{aligned}$$

In evaluating the right-hand side, we will use the formula

$$D_m h = -\overline{m_1 h}$$

obtained as follows: Let $\widetilde{h} = \int_0^\theta h_\theta d\theta$. Then,

$$h = \widetilde{h} - \int_\Delta \widetilde{h} ds, \qquad D_m h = -\int \widetilde{h} m_1 ds = -\int_\Delta h m_1 ds$$

Let m, h, k, p be tangent vector fields on $\widetilde{\Omega}$ which are sections of $T^0\widetilde{\Omega}$ and which are constant on Ω.

$$\begin{aligned}D^2_{m,p}\langle h,k\rangle &= -D_p\int_{S^1} h_\theta\cdot k_\theta m_1 e^{-z_1}d\theta\\ &= \int_{S^1} h_\theta\cdot k_\theta m_1 p_1 e^{-z_1}d\theta + \overline{m_1p_1}\int_{S^1} h_\theta\cdot k_\theta e^{-z_1}d\theta\\ &= \int_\Delta m_1p_1 h_s\cdot k_s ds + \overline{m_1p_1}\langle h,k\rangle\end{aligned}$$

We may assume that m, h are orthonormal at ζ. Then,

$$\begin{aligned} D^2_{m,h}\langle m,h\rangle &= \int_\Delta m_1 h_1 m_s\cdot h_s ds \\ -\frac{1}{2}D^2_{m,m}\langle h,h\rangle &= -\frac{1}{2}\int_\Delta m_1^2 h_s\cdot h_s ds - \frac{1}{2}\overline{m_1^2} \\ -\frac{1}{2}D^2_{h,h}\langle m,m\rangle &= -\frac{1}{2}\int_\Delta h_1^2 m_s\cdot m_s ds - \frac{1}{2}\overline{h_1^2} \end{aligned}$$

The expression $\langle\Gamma(m,h),\Gamma(m,h)\rangle - \langle\Gamma(m,m),\Gamma(h,h)\rangle$ simplifies to

$$\begin{aligned} &\frac{1}{2}\int_\Delta |h_1 m_s - m_1 h_s|^2 ds + \\ &\frac{1}{4}\int_\Delta \left[\left|D_s^{-1}(m_s\cdot h_s)^0\right|^2 - D_s^{-1}(m_s\cdot m_s)^0 D_s^{-1}(h_s\cdot h_s)^0\right] ds \\ &-\frac{1}{4}\left(\left|\overline{h_1 m_s} + \overline{m_1 h_s}\right|^2 + \overline{m_1^2} + \overline{h_1^2}\right) + \overline{m_1 m_s}\cdot\overline{h_1 h_s} \end{aligned}$$

Summing all the terms, we get,

$$\begin{aligned} &\kappa_{\zeta,\Omega}(m,h) \\ = &\frac{1}{4}\int_\Delta \left[\left|D_s^{-1}(m_s\cdot h_s)^0\right|^2 - D_s^{-1}(m_s\cdot m_s)^0 D_s^{-1}(h_s\cdot h_s)^0\right] ds \\ &-\frac{1}{4}\left(\left|\overline{h_1 m_s} + \overline{m_1 h_s}\right|^2 + 3\left(\overline{m_1^2} + \overline{h_1^2}\right)\right) + \overline{m_1 m_s}\cdot\overline{h_1 h_s} \end{aligned} \tag{5.2}$$

where m, h are orthonormal.

5.6. **Sectional Curvature of Ω_0.** Now let $\zeta \in \Omega_0$. Since Ω_0 is a submanifold of Ω, its sectional curvature may be calculated using Gauss Lemma.

5.6.1. *The Normal Bundle.* Let $z \in \widetilde{\Omega}$.

$$Q = Q_1 + iQ_2 = \int_0^{2\pi} e^{z_1 + iz_2} dx$$

$z \in \widetilde{\Omega_0}$ if and only if $Q = 0$. Now assume that $z \in \widetilde{\Omega_0}$ such that $\chi(z) = \zeta$. Let $h \in T_z^0\widetilde{\Omega}$.

$$\begin{aligned} D_h Q &= \int_0^{2\pi} e^{z_1 + iz_2}(h_1 + ih_2) dx \\ &= \int_\Delta h e^{iz_2} ds \end{aligned}$$

Let

$$u_1 = (\cos z_2, -\sin z_2), u_2 = (\sin z_2, \cos z_2)$$

Then, in vector notation,

$$\begin{aligned} D_h Q &= \int_\Delta (h\cdot u_1, h\cdot u_2) ds \\ &= -\int_\Delta (h_s\cdot(D_s^{-2}u_1)_s, h_s\cdot(D_s^{-2}u_2)_s) ds \end{aligned}$$

Therefore, the gradient

$$\nabla Q_i = -D_s^{-2}u_i \in T_z^0\widetilde{\Omega}$$

As a complex valued function

$$D_s^{-1}u_1 = \left(D_s^{-1}e^{iz_2}\right)^* = \left(c_z^0\right)^*$$

where the superscript $*$ indicates complex congugation. c_z^0 is the curve c_z translated so that its center of gravity is at the origin. Similarly, $D_s^{-1}u_2 = i\left(c_z^0\right)^*$ which is just a rotation of $\left(c_z^0\right)^*$ by $\pi/2$.

Clearly, $\nabla Q_1, \nabla Q_2$ are orthogonal. The normal vector space defined by the gradients ∇Q_1 and ∇Q_2 is invariant under change in z_2 by an additive constant and hence defines the normal vector space at $\zeta \in \Omega_0$. A choice of z correponds to a choice of a basis of the normal vector space at ζ. For $i = 1, 2$,

$$\|\nabla Q_i\|^2 = \langle \nabla Q_i, \nabla Q_i \rangle = \int_\Delta \left|c_z^0\right|^2 ds$$

$\|\nabla Q_i\|^2$ is the polar moment of the curve c_z^0, invariant under rotation of c_z^0 and hence, invariant under change in z_2 by an additive constant.

5.6.2. *The Second Fundamental Form.* The second fundamental form $S(m,h)$ has the decomposition:

$$S(m,h) = \frac{\langle S(m,h), \nabla Q_1\rangle \nabla Q_1}{\|\nabla Q_1\|^2} + \frac{\langle S(m,h), \nabla Q_2\rangle \nabla Q_2}{\|\nabla Q_2\|^2}$$

$$\langle S(m,h), \nabla Q_i\rangle = -\langle D_m \nabla Q_i, h\rangle = -\nabla^2 Q_i(m,h)$$

$$\text{The Hessian } \nabla^2 Q_i(m,h) = D_m D_h Q_i - D_{\Gamma(m,h)} Q_i$$

$$\begin{aligned} D_m D_h Q &= D_m \int_0^{2\pi} h e^z dx \\ &= \overline{m_1 h} \int_0^{2\pi} e^z dx + \int_0^{2\pi} m h e^z dx \\ &= \int_\Delta m h e^{iz_2} ds \text{ at } z \in \widetilde{\Omega_0} \end{aligned}$$

since, at $z \in \widetilde{\Omega_0}$, $\int_0^{2\pi} e^z dx = 0$.

$$D_{\Gamma(m,h)} Q = \int_\Delta \Gamma(m,h) e^{iz_2} ds$$

Therefore,

$$\langle S(m,h), \nabla Q\rangle = -\int_\Delta (mh - \Gamma(m,h))\, e^{iz_2} ds$$

$$\langle S(m,h), \nabla Q_i\rangle = \langle \Gamma(m,h) - (mh), \nabla Q_i\rangle$$

Therefore,

$$\begin{aligned} S(m,h) &= \Gamma(m,h) - (mh) \\ \Gamma(m,h) &= -\frac{1}{2} D_s^{-1}\left[(m_1 h_s)^0 + (h_1 m_s)^0 + \left(D_s^{-1}(m_s \cdot h_s)^0, 0\right)\right] \\ (mh) &= (m_1 h_1 - m_2 h_2, m_1 h_2 + m_2 h_1) \end{aligned}$$

5.6.3. *Gauss Lemma.*

$$\begin{aligned} & \kappa_{\zeta,\Omega_0}(m,h) \\ = \;& \kappa_{\zeta,\Omega}(m,h) + \langle S(m,m), S(h,h)\rangle - \langle S(m,h), S(m,h)\rangle \\ = \;& \kappa_{\zeta,\Omega}(m,h) + \left\langle \Gamma(m,m) - \left(m^2\right), \Gamma(h,h) - \left(h^2\right)\right\rangle \\ & - \langle \Gamma(m,h) - (mh), \Gamma(m,h) - (mh)\rangle \end{aligned}$$

This may be simplified to take the form

$$\kappa_{\zeta,\Omega_0}(m,h) = -3\left\|h_1 m_s - m_1 h_s\right\|_0^2 - \frac{1}{2}\left\|h_2 m_s - m_2 h_s\right\|_0^2 - 9\overline{(h_1 m_2 - m_1 h_2)(h_{1s} m_{2s} - m_{1s} h_{2s})} - \frac{1}{2}\left(\overline{m_1^2} + \overline{h_1^2}\right) \tag{5.3}$$

where m, h are orthonormal.

5.7. **Sectional Curvature of $\Omega_0/\operatorname{Diff}^+(S^1)$.** Let $\zeta \in \Omega_0$. Let $z \in \widetilde{\Omega_0}$ map to ζ. Let a, b be a pair of orthonormal tangent vectors at $\Pi(\zeta) \in \Omega_0/\operatorname{Diff}^+(S^1)$. Let m, h be orthonormal horizontal lifts of a, b in $T^0_z\widetilde{\Omega_0}$. Let $m^\#, h^\#$ be sections of $T^0\widetilde{\Omega_0}$ in a neighborhood of z which are horizontal extensions of m, h. Then, by O'Neill's formula [7], we have

$$\kappa_{\Pi(\zeta),\Omega_0/\operatorname{Diff}^+(S^1)}(a,b) = \kappa_{\zeta,\Omega_0}(m,h) + \frac{3}{4}\left\|\left[m^\#_\theta, h^\#_\theta\right]^V_{\zeta,\Omega_0}\right\|^2 \tag{5.4}$$

where $\left[m^\#_\theta, h^\#_\theta\right]$ is the Lie bracket and as before, the superscript V denotes its Π-vertical component. We now derive an explicit expression for $\left[m^\#_\theta, h^\#_\theta\right]^V_{\zeta,\Omega_0}$.

We may regard the vector fields $m^\#, h^\#$ as vector fields on $\widetilde{\Omega}$ since $\left[m^\#_\theta, h^\#_\theta\right]_{\zeta,\Omega_0} = \left[m^\#_\theta, h^\#_\theta\right]_{\zeta,\Omega}$. We now construct the vector fields $m^\#, h^\#$. An important fact implicit in O'Neill's formula is that $\left[m^\#_\theta, h^\#_\theta\right]^V_{\zeta,\Omega}$ is independent of the choice of the horizontal extensions $m^\#, h^\#$. Extend the vectors m, h as sections of $T^0\widetilde{\Omega}$ such that m_θ, h_θ are constant vector fields on Ω and denote them again as m, h respectively. Set $m^\# = m^H$ and $h^\# = h^H$. Since $D_{m_\theta} h_\theta = D_{h_\theta} m_\theta = 0$,

$$\left[m^\#_\theta, h^\#_\theta\right]^V_{\zeta,\Omega_0} = D_{m_\theta} h^H_\theta - D_{h_\theta} m^H_\theta = D_{h_\theta} m^V_\theta - D_{m_\theta} h^V_\theta$$

Let $D_{\zeta,h} f$ denote the derivative of f at the point ζ in the direction h.

$$\begin{aligned} D_{\zeta,h_\theta} m^V_\theta &= -\left(D_{\zeta,h}\left(L_1 L^{-1}\left(L_1^* \cdot m_{ss}\right)\right)\right)_\theta \\ &= -\left(L_1 L^{-1} D_{\zeta,h}\left(L_1^* \cdot m_{ss}\right)\right)_\theta \end{aligned}$$

since $L_1^* \cdot m_{ss} = 0$ at ζ. Therefore,

$$\begin{aligned} \left[m^\#_\theta, h^\#_\theta\right]^V_{\zeta,\Omega_0} &= \left(L_1 L^{-1}\psi\right)_\theta \\ \text{where} \quad \psi &= D_{\zeta,m}\left(L_1^* \cdot h_{ss}\right) - D_{\zeta,h}\left(L_1^* \cdot m_{ss}\right) \end{aligned}$$

We use the following formulae:

$$\begin{aligned} D_{\zeta,h} m_s &= D_{\zeta,h} e^{-z_1} m_\theta = -h_1 m_s \\ D_{\zeta,h} m_{ss} &= -2h_1 m_{ss} - h_{1s} m_s \\ D_{\zeta,h} \kappa &= D_{\zeta,h} z_{2s} \\ &= -h_1 z_{2s} + h_{2s} \\ &= -h_1 \kappa + h_{2s} \\ D_{\zeta,h} L_1^* &= -h_1 L_1^* + (0, h_{2s}) \end{aligned}$$

We recall that κ without subscripts denotes the curvature of c_z and equals z_{2s}. Since $L_1^* \cdot m_{ss} = 0$ at ζ,

$$\begin{aligned} D_{\zeta,h} \left(L_1^* \cdot m_{ss} \right) &= \left(D_{\zeta,h} L_1^* \right) \cdot m_{ss} + L_1^* \cdot D_{\zeta,h} m_{ss} \\ &= h_{2s} m_{2ss} - 2L_1^* \cdot (h_1 m_{ss}) - L_1^* \cdot (h_{1s} m_s) \\ &= h_{2s} m_{2ss} + 2h_{1s} m_{1ss} + (h_{1s} m_{1s})_s - \kappa \left(h_{1s} m_{2s} \right) \end{aligned}$$

taking into account again that $L_1^* \cdot m_{ss} = 0$ at z. We get

$$\begin{aligned} \psi = &\left(m_{2s} h_{2ss} - h_{2s} m_{2ss} \right) \\ &+2 \left(m_{1s} h_{1ss} - h_{1s} m_{1ss} \right) + \kappa \left(h_{1s} m_{2s} - m_{1s} h_{2s} \right) \end{aligned}$$

$$\begin{aligned} \left\| \left[m_\theta^\#, h_\theta^\# \right]_{\zeta,\Omega_0}^V \right\|^2 &= \int_\Delta \left(L_1 L^{-1} \psi \right)_s \cdot \left(L_1 L^{-1} \psi \right)_s ds \\ &= \int_\Delta \left(L^{-1} \psi \right) \left[\left(L^{-1} \psi \right)_{ssss} - \kappa \left(\kappa L^{-1} \psi \right)_{ss} \right] ds \\ &= \int_\Delta \psi L^{-1} \psi ds \end{aligned}$$

$$\kappa_{\Pi(\zeta),\Omega_0/\operatorname{Diff}^+(S^1)}(a,b) = \kappa_{\zeta,\Omega_0}(m,h) + \frac{3}{4} \int_\Delta \psi L^{-1} \psi ds \tag{5.5}$$

where a, b are orthonormal.

5.8. **Absolute Bounds.** We derive the following bounds for the sectional curvature.

Let $\zeta \in \Omega$.

$$|\kappa_{\zeta,\Omega}| \le \frac{9}{2} + \frac{19}{8\pi^2} \tag{5.6}$$

If $\zeta \in \Omega_0$, then,

$$|\kappa_{\zeta,\Omega_0}| \le 72 + \frac{1}{2\pi^2} \tag{5.7}$$

Let $z \in \widetilde{\Omega}$ map to $\zeta \in \Omega$. Let a, b be a pair of orthonormal tangent vectors at $\Pi(\zeta) \in \Omega_0/\operatorname{Diff}^+(S^1)$. Let m, h be orthonormal horizontal lifts of a, b in $T_z^0 \widetilde{\Omega_0}$. Fix h. Then, for all a,

$$(5.8)\qquad \begin{aligned} &\left|\kappa_{\Pi(\zeta),\Omega_0/\operatorname{Diff}^+(S^1)}(a,b)\right| \\ &\leq 72+\frac{1}{2\pi^2}+\frac{3}{4}\left(\frac{3}{8\pi}+\sqrt{\left(\frac{3}{8\pi}\right)^2+C}\right)^2\left(6\left\|h\right\|_2+2\left\|\kappa\right\|_{L^\infty}\right)^2 \end{aligned}$$

where

$$C=\frac{1+\frac{3\sqrt{3}}{\pi}\left\|\kappa\right\|_{L^\infty}\left\|\kappa\right\|_2\left(1+\frac{2\pi J}{\left\|\kappa\right\|_{L^\infty}}+\frac{2\pi J}{4\sqrt{3}\left\|\kappa\right\|_2}\right)}{2\pi^2\left(1-\overline{\kappa A^{-1}\kappa}\right)}$$

κ is the curvature of the curve c_z corresponding to the point z and J is its rotation index.

5.8.1. *Sobolev Spaces.* Let $C^{\infty,0}(\Delta,\mathbb{R})=\left\{f\in C^\infty(\Delta,\mathbb{R}):\overline{f}=0\right\}$. Define norms $\left\|f\right\|_n$, $n\in\mathbb{Z}$, by setting

$$(5.9)\qquad \left\|f\right\|_s=\left[\int_\Delta\left|D_s^n f\right|^2 ds\right]^{\frac{1}{2}}$$

Let $H^n(\Delta,\mathbb{R})$ denote the completion of $C^{\infty,0}(\Delta,\mathbb{R})$ in the norm $\left\|\cdot\right\|_n$. If $(f_1,f_2)\in C^{\infty,0}(\Delta,\mathbb{R}^2)$, its norm

$$\left\|(f_1,f_2)\right\|_n=\left[\int_\Delta\left(\left|D_s^n f_1\right|^2+\left|D_s^n f_2\right|^2\right)ds\right]^{\frac{1}{2}}$$

defines the corresponding space $H^n(\Delta,\mathbb{R}^2)$. If $f\in H^n(\Delta,\mathbb{R})$ and $g\in H^{-n}(\Delta,\mathbb{R})$, $fg\in L^1(\Delta,\mathbb{R})$:

$$(5.10)\qquad \int_\Delta|fg|ds\leq\left\|f\right\|_n\left\|g\right\|_{-n}$$

The following estimate may be proved by means of the Fourier series of $f\in H^n(\Delta,\mathbb{R})$:

$$(5.11)\qquad \left\|f\right\|_m\leq\frac{\left\|f\right\|_n}{(2\pi)^{n-m}},\ m<n$$

For $n>0$, $H^n(\Delta,\mathbb{R})$ is an algebra in which the multiplication is defined as $(fg)^0$.

$$(5.12)\qquad \left\|(fg)^0\right\|_1 \leq \sqrt{2}\left\|f\right\|_1\left\|g\right\|_1$$

$$(5.13)\qquad \left\|(fg)^0\right\|_n \leq \frac{4\sqrt{3}\left\|f\right\|_n\left\|g\right\|_n}{\pi^{n-2}} \text{ if } n>1$$

In what follows, we will refer to the function spaces defined above simply as H^n, L^1, etc. and omit the specification $(\Delta,\mathbb{R})$ or $(\Delta,\mathbb{R}^2)$.

5.8.2. *An upper bound for* $|\kappa_{\zeta,\Omega}|$.

$$\begin{aligned} & \kappa_{\zeta,\Omega}(m,h) \\ = & \frac{1}{4}\int_\Delta \left[\left|D_s^{-1}\left(m_s\cdot h_s\right)^0\right|^2 - D_s^{-1}\left(m_s\cdot m_s\right)^0 D_s^{-1}\left(h_s\cdot h_s\right)^0\right] ds \\ & -\frac{1}{4}\left(\left|\overline{h_1 m_s}+\overline{m_1 h_s}\right|^2 + 3\left(\overline{m_1^2}+\overline{h_1^2}\right)\right) + \overline{m_1 m_s}\cdot\overline{h_1 h_s} \end{aligned}$$

where m, h are orthonormal. We estimate below typical terms in the expression.

If $f \in L^1$,

$$|\int_0^s f^0 ds \le \left|\int_0^s f ds\right| + |s\overline{f}| \quad \text{for } -\frac{1}{2}\le s \le \frac{1}{2}$$

It follows that $|D_s^{-1}(f^0)| \le \frac{3}{2}\|f\|_{L^1}$ and hence, $\left\|f^0\right\|_{-1} \le \frac{3}{2}\|f\|_{L^1}$.

Since m and h are unit vectors,

$$\begin{aligned} \left|D_s^{-1}\left(m_s\cdot h_s\right)^0\right|^2 &\le 9 \\ \left|D_s^{-1}\left(m_s\cdot m_s\right)^0 D_s^{-1}\left(h_s\cdot h_s\right)^0\right| &\le 9 \end{aligned}$$

$$\overline{m_1^2} = \|m_1\|_0^2 \le \left(\frac{\|m_1\|_1}{2\pi}\right)^2 \le \frac{1}{4\pi^2}$$

$$|\overline{h_1 m_{1s}}| \le \|h_1\|_0 \left(\|m_1\|_1 + \|m_1\|_1\right) \le \frac{1}{\pi}$$

Substituting these inequalities in the expression for $\kappa_{\zeta,\Omega}$, we get

$$|\kappa_{\zeta,\Omega}| \le \frac{9}{2} + \frac{19}{8\pi^2} \tag{5.14}$$

5.8.3. *An upper bound for* $|\kappa_{\zeta,\Omega_0}|$.

$$\begin{aligned} \kappa_{\zeta,\Omega_0}(m,h) = & -3\left\|h_1 m_s - m_1 h_s\right\|_0^2 - \frac{1}{2}\left\|h_2 m_s - m_2 h_s\right\|_0^2 \\ & -9\overline{\left(h_1 m_2 - m_1 h_2\right)\left(h_{1s} m_{2s} - m_{1s} h_{2s}\right)} \\ & -\frac{1}{2}\left(\overline{m_1^2}+\overline{h_1^2}\right) \end{aligned}$$

where m, h are orthonormal.

$$\|h_1 m_s\|_0 \le \|h_1\|_{L^\infty}\|m_s\|_0 \le \frac{3}{2}\|h_1 m_s - m_1 h_s\|_0 \le 3$$

$$|h_1 m_2 - m_1 h_2| \le \frac{9}{2}$$

$$\begin{aligned} & \left|\overline{\left(h_1 m_2 - m_1 h_2\right)\left(h_{1s} m_{2s} - m_{1s} h_{2s}\right)}\right| \\ \le & \frac{9}{2}\left|\left(\overline{h_{1s} m_{2s} - m_{1s} h_{2s}}\right)\right| \le \frac{9}{2}\overline{|h_s|\,|m_s|} \\ \le & \frac{9}{2}\left(\|h_s\|_0\|m_s\|_0\right) \le \frac{9}{2} \end{aligned}$$

$$\begin{aligned} |\kappa_{\zeta,\Omega_0}| &\le 3\cdot 9 + \frac{9}{2} + 9\cdot\frac{9}{2} + \frac{1}{2\pi^2} \\ &\le 72 + \frac{1}{2\pi^2} \end{aligned} \tag{5.15}$$

5.8.4. *An upper bound for* $|\kappa_{\Pi(\zeta),\Omega_0/\,\mathrm{Diff}^+(S^1)}|$.

$$\kappa_{\Pi(\zeta),\Omega_0/\,\mathrm{Diff}^+(S^1)}(a,b) = \kappa_{\zeta,\Omega_0}(m,h) + \frac{3}{4}\int_\Delta \psi L^{-1}\psi ds$$

where

$$\begin{aligned}\psi &= (m_{2s}h_{2ss} - h_{2s}m_{2ss}) \\ &\quad +2\,(m_{1s}h_{1ss} - h_{1s}m_{1ss}) + \kappa\,(h_{1s}m_{2s} - m_{1s}h_{2s})\end{aligned}$$

and m, h are orthonormal.

$$\begin{aligned}\overline{\psi L^{-1}\psi} &= \int_\Delta \left(L^{-1}\psi\right) L \left(L^{-1}\psi\right) ds \\ &= \int_\Delta \left[\left(L^{-1}\psi\right)_{ss}^2 + \left(\kappa L^{-1}\psi\right)_s^2\right] ds\end{aligned}$$

For $\overline{\psi L^{-1}\psi}$ to be finite, $L^{-1}\psi \in H^2$ or $\psi \in H^{-2}$. Orthonormality of m, h implies that $h_s, m_s \in H^0$ and $h_{ss}, m_{ss} \in H^{-1}$. This is not sufficient to place the terms like $m_{2s}h_{2ss}$ in H^{-2}. We have to assume additional regularity at least for one of the tangent vectors m, h. We assume that $h \in H^2$. We fix h and derive an upper bound with respect to m. There is an additional complication because $\overline{\psi}$ need not be equal to zero. We will see that under this assumption, $\psi \in L^1$. Therefore,

$$\begin{aligned}|\overline{\psi}| &\leq \|\psi\|_{L^1} \\ \left\|\psi^0\right\|_{-2} &\leq \frac{1}{2\pi}\left\|\psi^0\right\|_{-1} \leq \frac{1}{2\pi}\cdot\frac{3}{2}\|\psi\|_{L^1}\end{aligned}$$

Let $u = L^{-1}\psi$ so that $\overline{\psi L^{-1}\psi} = \overline{u\psi} = \overline{uLu}$. Since $\overline{u\psi} = \overline{u}\,\overline{\psi} + \overline{u^0\psi^0}$,

$$\overline{\psi L^{-1}\psi} \leq |\overline{u}|\,|\overline{\psi}| + \left\|u^0\right\|_2 \left\|\psi^0\right\|_{-2}$$

Since $\overline{uLu} = \overline{u_{ss}^2 + (\kappa u)_s^2} \geq \overline{(u_{ss}^0)^2}$, $\left\|u^0\right\|_2 \leq \sqrt{\overline{\psi L^{-1}\psi}}$. We get

$$\begin{aligned}\overline{\psi L^{-1}\psi} &\leq |\overline{u}|\,|\overline{\psi}| + \sqrt{\overline{\psi L^{-1}\psi}}\left\|\psi^0\right\|_{-2} \\ &\leq |\overline{u}|\,\|\psi\|_{L^1} + \frac{3}{4\pi}\sqrt{\overline{\psi L^{-1}\psi}}\,\|\psi\|_{L^1}\end{aligned}$$

$$\overline{\psi L^{-1}\psi} \leq \left(\frac{3}{8\pi} + \sqrt{\left(\frac{3}{8\pi}\right)^2 + \frac{|\overline{u}|}{\|\psi\|_{L^1}}}\right)^2 \|\psi\|_{L^1}^2 \qquad (5.16)$$

We now proceed to estimate $\overline{u}$, and $\|\psi\|_{L^1}$.

Let $L_0 = D_s^4 - 4\pi^2 J^2 D_s^2$ where J is the rotation index. We have $\kappa = \kappa^0 + 2\pi J$. Let $M = L_0 - L$ so that $Mb = 2\pi J\left((\kappa^0 b)_{ss} + \kappa^0 b_{ss}\right) + \kappa^0(\kappa^0 b)_{ss}$. Let $v = L_0^{-1}\psi^0$ and let $w = L^{-1}\overline{\psi}$. Note that $4\pi^2 J^2 v = D_s^2 v - D_s^{-2}\psi^0$ so that $\overline{v} = 0$. Let $\psi_* = Mv$. Let $u_* = L^{-1}\psi_*$. We have

$$\begin{aligned}u &= w + v + u_* \\ \overline{u} &= \overline{w} + \overline{u_*}\end{aligned}$$

Let λ be the smallest eigenvalue of L.

$$\begin{aligned}\lambda \|w\|_0^2 &\leq \overline{wLw} \leq |\overline{w}|\,|\overline{\psi}| \leq |\overline{\psi}|\,\|w\|_0 \\ \text{Therefore, } \|w\|_0 &\leq \frac{|\overline{\psi}|}{\lambda}\end{aligned}$$

Note that the norms $\|\cdot\|$ extend to $C^\infty(\Delta,\mathbb{R})$ if $n \geq 0$. Since $|\overline{w}| \leq \|w\|_0$,

$$|\overline{w}| \leq \frac{|\overline{\psi}|}{\lambda}$$

Since $\psi \in H^{-2}$, $\psi^0 \in H^{-2}$, $u, v \in H^2$, $\psi_* \in H^0$ and $u_* \in H^4$.

$$\begin{aligned}\lambda \|u_*\|_0^2 &\leq \overline{u_* L u_*} \leq \overline{u_* \psi_*} \leq \|u_*\|_0 \|\psi_*\|_0 \\ |\overline{u_*}| &\leq \|u_*\|_0 \leq \frac{\|\psi_*\|_0}{\lambda}\end{aligned}$$

It follows that

$$|\overline{u}| \leq \frac{1}{\lambda}\left(|\overline{\psi}| + \|\psi_*\|_0\right) \leq \frac{1}{\lambda}\left(1 + \frac{\|\psi_*\|_0}{\|\psi\|_{L^1}}\right)\|\psi\|_{L^1} \tag{5.17}$$

Let

$$\begin{aligned}D_s^{-2}\psi^0 &= \sum_{k\neq 0} a_k e^{2\pi iks} \\ v &= \sum_{k\neq 0} b_k e^{2\pi iks}\end{aligned}$$

Substituting these in the equation $v_{ss} - 4\pi^2 J^2 v = D_s^{-2}\psi^0$, we get

$$\begin{aligned}b_k &= -\frac{a_k}{4\pi^2(k^2+J^2)} \\ v_{ss} &= \sum_{k\neq 0} \frac{k^2}{k^2+J^2} a_k e^{2\pi iks}\end{aligned}$$

$$\begin{aligned}\|v\|_2 &= \|v_{ss}\|_0 = \left[\sum_{k\neq 0}\left(\frac{k^2}{k^2+J^2}a_k\right)^2\right]^{1/2} \\ &\leq \left[\sum_{k\neq 0} a_k^2\right]^{1/2} \leq \left\|\psi^0\right\|_{-2}\end{aligned}$$

$\psi_* = 2\pi J\left(\left(\kappa^0 v\right)_{ss} + \kappa^0 v_{ss}\right) + \kappa^0\left(\kappa^0 v\right)_{ss}$.

$$\begin{aligned}\left\|\left(\kappa^0 v\right)_{ss}\right\|_0 &= \left\|\kappa^0 v\right\|_2 \leq 4\sqrt{3}\left\|\kappa^0\right\|_2 \|v\|_2 \\ &\leq 4\sqrt{3}\,\|\kappa\|_2 \left\|\psi^0\right\|_{-2}\end{aligned}$$

$$\left\|\kappa^0 v_{ss}\right\|_0 \leq \|\kappa\|_{L^\infty}\left\|\psi^0\right\|_{-2}$$

$$\begin{aligned}\left\|\kappa^0\left(\kappa^0 v\right)_{ss}\right\|_0 &\leq \|\kappa\|_{L^\infty}\left\|\left(\kappa^0 v\right)_{ss}\right\|_0 \\ &\leq 4\sqrt{3}\,\|\kappa\|_{L^\infty}\|\kappa\|_2\left\|\psi^0\right\|_{-2}\end{aligned}$$

We also have $\left\|\psi^0\right\|_{-2} \leq \frac{3}{4\pi}\left\|\psi\right\|_{L^1}$. We get

$$\left\|\psi_*\right\|_0 \leq \frac{3\sqrt{3}}{\pi}\left\|\kappa\right\|_{L^\infty}\left\|\kappa\right\|_2\left(1+\frac{2\pi J}{\left\|\kappa\right\|_{L^\infty}}+\frac{2\pi J}{4\sqrt{3}\left\|\kappa\right\|_2}\right)\left\|\psi\right\|_{L^1} \tag{5.18}$$

Estimates for typical terms in the expression for ψ are as follows.

$$\begin{aligned}\left\|m_{2s}h_{2ss}\right\|_{L^1} &\leq \left\|m_{2s}\right\|_0\left\|h_{2ss}\right\|_0 \leq \left\|h\right\|_2\\ \left\|m_{2ss}h_{2s}\right\|_{L^1} &\leq \left\|m_{2ss}\right\|_{-1}\left\|h_{2s}\right\|_1 \leq \left\|h\right\|_2\\ \left\|\kappa m_{1s}h_{2s}\right\|_{L^1} &\leq \left\|\kappa\right\|_{L^\infty}\left\|m_{1s}\right\|_0\left\|h_{2s}\right\|_0 \leq \left\|\kappa\right\|_{L^\infty}\end{aligned}$$

Therefore,

$$\left\|\psi\right\|_{L^1} \leq 6\left\|h\right\|_2 + 2\left\|\kappa\right\|_{L^\infty} \tag{5.19}$$

Substituting the estimates (5.17), (5.18), (5.19) and a lower bound (5.20) for λ derived below in the inequality (5.16), we obtain an estimate for O'Neill's bracket.

5.8.5. *A Lower Bound for λ.* In this section, H^n denotes the completion of $C^\infty(\Delta,\mathbb{R})$ or $C^\infty(\Delta,\mathbb{R}^2)$ in the norm $\left\|\cdot\right\|_n$.

Recall that $L_1 = (D_s,\kappa)$. For $b, b_1, b_2 \in H^2$,

$$\begin{aligned}\left\langle b_1, b_2\right\rangle_L &= \int_\Delta b_1 L b_2 ds = \int_\Delta \left(L_1 b_1\right)_s \cdot \left(L_1 b_2\right)_s ds\\ \left\|b\right\|_L^2 &= \left\langle b, b\right\rangle_L = \int_\Delta \left(L_1 b\right)_s \cdot \left(L_1 b\right)_s ds = \left\|L_1 b\right\|_1^2\end{aligned}$$

$$\lambda = \min_{b\neq 0}\frac{\left\|b\right\|_L^2}{\left\|b\right\|_0^2} = \min_{b\neq 0}\frac{\left\|L_1 b\right\|_1^2}{\left\|b\right\|_0^2}$$

Let A be the positive definite operator $-D_s^2+\kappa^2$. Let

$$\begin{aligned}\left\langle b_1, b_2\right\rangle_A &= \int_\Delta b_1 A b_2 ds = \int_\Delta \left(L_1 b_1\right)\cdot\left(L_1 b_2\right) ds\\ \left\|b\right\|_A^2 &= \left\langle b, b\right\rangle_A = \int_\Delta \left(L_1 b\right)\cdot\left(L_1 b\right) ds = \left\|L_1 b\right\|_0^2\end{aligned}$$

We have the inequality

$$\left\|\left(L_1 b\right)^0\right\|_0^2 \leq \frac{\left\|L_1 b\right\|_1^2}{4\pi^2}$$

and the following estimate from [1]:

$$\left\|\left(L_1 b\right)\right\|_0^2 = \left\|b\right\|_A^2 \geq \frac{1}{2}\left\|b\right\|_0^2$$

Therefore,

$$\lambda \geq \frac{2\pi^2\left\|\left(L_1 b\right)^0\right\|_0^2}{\left\|\left(L_1 b\right)\right\|_0^2}$$

We need to relate $\left\|\left(L_1 b\right)^0\right\|_0$ and $\left\|L_1 b\right\|_0$.

Let $U = (0,1) \in T_z\widetilde{\Omega}$. Then, $L_1 b = \left(L_1 b\right)^0 + \overline{\kappa b}U$. $L_1 b = \left(L_1 b\right)^0$ if and only if

$$0 = \int_\Delta \kappa b ds = \int_\Delta bA\left(A^{-1}\kappa\right) ds = \left\langle b, A^{-1}\kappa\right\rangle_A$$

Let $\beta = A^{-1}\kappa$. $\overline{\kappa\beta} = \|\beta\|_A^2 = \overline{(\beta_s^2 + \kappa^2\beta^2)} \geq \|\kappa\beta\|_0^2$. We also have $\overline{\kappa\beta} \leq \|\kappa\beta\|_0$ by Schwartz inequality. Therefore, $\overline{\kappa\beta} \leq \left(\overline{\kappa\beta}\right)^{1/2}$ and hence, $\|\beta\|_A \leq 1$. Equality requires β to be constant which, in turn, implies that κ is constant, that is, c_z is a circle. Write

$$b = b_0 + a\beta \text{ where } a = \frac{\langle b, \beta\rangle_A}{\|\beta\|_A^2}$$

Then,

$$\|(L_1 b)\|_0^2 = \|b\|_A^2 = \|b_0\|_A^2 + a^2\,\|\beta\|_A^2$$

and since $L_1\beta = (L_1\beta)^0 + \overline{\kappa\beta}U = (L_1\beta)^0 + \|\beta\|_A^2\,U$,

$$\begin{aligned} (L_1 b)^0 &= (L_1 b_0) + a\,(L_1\beta)^0 = (L_1 b_0) + aL_1\beta - a\,\|\beta\|_A^2\,U \\ \left\|(L_1 b)^0\right\|_0^2 &= \|b_0\|_A^2 + a^2\,\|\beta\|_A^2\left(1 - \|\beta\|_A^2\right) \end{aligned}$$

Therefore,

$$\frac{\left\|(L_1 b)^0\right\|_0^2}{\|(L_1 b)\|_0^2} = \frac{\|b_0\|_A^2 + a^2\,\|\beta\|_A^2\left(1 - \|\beta\|_A^2\right)}{\|b_0\|_A^2 + a^2\,\|\beta\|_A^2} \geq 1 - \|\beta\|_A^2 \geq 1 - \overline{\kappa A^{-1}\kappa}$$

and

$$\lambda \geq 2\pi^2\left(1 - \overline{\kappa A^{-1}\kappa}\right) \tag{5.20}$$

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