

H^0 -TYPE RIEMANNIAN METRICS ON THE SPACE OF PLANAR CURVES

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ABSTRACT. Michor and Mumford have shown that the distances between planar curves in the simplest metric (not involving derivatives) are identically zero. We derive geodesic equations and a formula for sectional curvature for conformally equivalent metrics. We show if the conformal factor depends only on the length of the curve, the metric behaves like an L^1 metric, the sectional curvature is not bounded from above and minimal geodesics may not exist. If the conformal factor is superlinear in curvature, the sectional curvature is bounded from above.

1. INTRODUCTION

The purpose of this paper is to study the most basic properties of some of the simplest Riemannian metrics suggested by applications to Computer Vision. The problem is to understand and quantify similarities and differences between object shapes and their individual variations. At a fundamental level, the problem is to construct appropriate metrics on a space of closed surfaces in \mathbb{R}^3 . A simpler version of the problem is the construction of Riemannian metrics on a space of closed planar curves. The choice of a metric depends on the type of similarity that is being considered. In their seminal paper [2], Michor and Mumford analyze two Riemannian metrics on a space of closed planar curves. Surprisingly, the Riemannian distance between any two curves in the simpler of the two metrics, an H^0 -metric, turns out to be zero. To remedy this, they add a curvature term to the metric. Below, we analyze conformal variants of the H^0 -metric of Michor and Mumford. After fixing the basic framework in §2, we investigate in §3 the existence of minimal geodesics in two specific cases where the conformal factor depends only on length of the curve. We derive upper and lower bounds for distances between curves and show that these metrics behave like L^1 metrics. In the case of simpler of the two metrics, the only minimal geodesics are those which deform the curve by moving all of its points with the same normal speed. In the case of the second metric, no geodesic is minimal if the length of the curve is less than a certain threshold; the question of minimality when the length of the deforming curve is equal or greater than threshold is still open. In §4, we derive an explicit geodesic equation and in §5, we provide a formula for the sectional curvature. We show that the sectional curvature is unbounded from above if the conformal factor depends only on the

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length of the curve. If the conformal factor is a superlinear function of the curvature, the sectional curvature at a point is bounded with respect to all the planes passing through a fixed tangent vector.

Conformal H^o metrics using conformal factors depending on the length of the curve were proposed by Yezzi and Mennucci in [7,8]. Higher order metrics have also been proposed. Younes has proposed an H^1 metric in [9]. Mio et al [4,5,6] have constructed geodesics in H^1 and H^2 metrics. More recently, Michor and Mumford have described a general Hamiltonian framework for studying Sobolev metrics on the space of planar curves [3].

2. THE FRAMEWORK

The basic space considered by Michor and Mumford is the orbit space

$$B_e(S^1, \mathbb{R}^2) = \text{Emb}(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

of the space of all C^∞ embeddings of S^1 in the plane, under the action by composition from the right by diffeomorphisms of the unit circle. It is contained in the bigger space of immersions modulo diffeomorphisms:

$$B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

Let $\pi : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow B_i(S^1, \mathbb{R}^2)$ be the canonical projection. The simpler of the two metrics considered in [2] is an H^o -metric defined on $\text{Imm}(S^1, \mathbb{R}^2)$:

$$G_c^o(m, h) = \int_{S^1} (m \cdot h) |c_\theta| d\theta$$

where $c : S^1 \rightarrow \mathbb{R}^2$ is an immersion, defining a point in $\text{Imm}(S^1, \mathbb{R}^2)$, $m, h \in C^\infty(S^1, \mathbb{R}^2)$ are the vector fields along the image curve, defining two tangent vectors on $\text{Imm}(S^1, \mathbb{R}^2)$ at c , and $c_\theta = dc/d\theta$. $(m \cdot h)$ is the usual dot product in \mathbb{R}^2 . Sometimes for the sake of clarity, we will use the notation $a \cdot b$ even when a, b are scalars. Let n_c denote the unit normal field along c . If we identify \mathbb{R}^2 with the complex plane \mathbb{C} , then, $n_c = ic_\theta / |c_\theta|$. The tangent vectors on $B_i(S^1, \mathbb{R}^2)$ at $\pi(c)$ are of the form an_c where $a \in C^\infty(S^1, \mathbb{R})$. For any $C_o, C_1 \in B_i$, consider all liftings c_o, c_1 to $\text{Imm}(S^1, \mathbb{R}^2)$ and all smooth paths $t \mapsto (\theta \mapsto c(t, \theta))$, $0 \leq t \leq 1$, in $\text{Imm}(S^1, \mathbb{R}^2)$ with $c(0, \cdot) = c_o$ and $c(1, \cdot) = c_1$. Let c_t denote $\partial c / \partial t$ and $c_t^\perp = (c_t \cdot n_c) n_c$. The arc-length of such a path c is given by

$$\int_0^1 \sqrt{G_c^o(c_t, c_t)} dt$$

Michor and Mumford show that for any two curves in $B_i(S^1, \mathbb{R}^2)$,

$$\text{dist}_{G^o}(C_1, C_2) =_{def} \inf_c \int_0^1 \sqrt{G_c^o(c_t^\perp, c_t^\perp)} dt = 0$$

and strengthen G^o by defining

$$G_c^A(m, h) = \int_{S^1} (1 + A\kappa_c^2) (m \cdot h) |c_\theta| d\theta$$

where κ_c is the curvature, defined by the equation $\left(\frac{c_\theta}{|c_\theta|}\right)_\theta = i\kappa_c c_\theta = \kappa_c |c_\theta| n_c$.

An alternative is to consider conformal transformations of G^o :

$$G_c^\rho(m, h) = \rho(c) \int_{S^1} (m \cdot h) |c_\theta| d\theta$$

where $\rho(c)$ is a $\text{Diff}(S^1)$ -invariant function on $\text{Imm}(S^1, \mathbb{R}^2)$. In this paper, we consider conformal factors ρ which are functions of the curve length ℓ and the curvature κ .

Notation: Throughout this paper, we will use the superscript ρ (respectively o) to label quantities which are calculated using the metric G^ρ (respectively G^o) except that $G_c^\rho(h, h)$ will be denoted as $\|h\|_\rho^2$ and $G^o(h, h)$ simply as $\|h\|^2$.

3. INSTABILITY OF $G^{\rho(\ell)}$ GEODESICS

In this section, we assume that ρ is a function of the curve length ℓ alone: $\rho(c) = \ell(c)$ or $e^{A\ell(c)}$ where A is a positive constant. If c is a path connecting curves C, C' , let $\alpha(c)$ denote the area swept out by c in \mathbb{R}^2 . For a path $c(t, \cdot)$, let $\ell_{\max}(c) = \max_t \ell(c(t, \cdot))$. The following theorem characterizes the L^1 -type behavior of the metrics $G^{\rho(\ell)}$.

Theorem 3.1. *If $\rho(c) = \ell(c)$, then,*

$$\text{dist}_{G^\rho}(C, C') = \inf_c \alpha(c)$$

If $\rho(c) = e^{A\ell(c)}$, then,

$$\inf_c \sqrt{Ae} \alpha(c) \leq \text{dist}_{G^\rho}(C, C') \leq \inf_c \sqrt{Ae} e^{A\ell_{\max}(c)/2} \alpha(c)$$

We first prove a series of lemmas.

Lemma 3.2.

$$\text{dist}_{G^\rho}(C, C') \geq \begin{cases} \inf_c \alpha(c) & \text{if } \rho(c) = \ell(c) \\ \inf_c \sqrt{Ae} \alpha(c) & \text{if } \rho(c) = e^{A\ell(c)} \end{cases}$$

Proof. For any path c ,

$$\begin{aligned} L_{G^\rho}(c) &= \int_0^1 \left[\rho(c) \int_{S^1} (c_t^\perp \cdot c_t^\perp) |c_\theta| d\theta \right]^{\frac{1}{2}} dt \\ &\geq \int_0^1 \left[\left(\frac{\rho(c)}{|\sup(c_t^\perp)|} \right)^{\frac{1}{2}} \int_{S^1} |c_t^\perp| |c_\theta| d\theta \right] dt \\ &\geq \left[\min_t \frac{\rho(c)}{\ell(c)} \right]^{\frac{1}{2}} \int_{S^1 \times [0,1]} |\det dc(t, \theta)| d\theta dt \\ &\geq \begin{cases} \alpha(c) & \text{if } \rho(c) = \ell(c) \\ \sqrt{Ae} \alpha(c) & \text{if } \rho(c) = e^{A\ell(c)} \end{cases} \end{aligned}$$

□

David Mumford observed from the formula for the sectional curvature that the geodesics along which $|\sup(c_t^\perp)| < \ell$ if $\rho = \ell$ and $< 1/A$ if $\rho = e^{A\ell}$ may not be minimal. Such a possibility can be heuristically seen from the inequality

$$L_{G^\rho}(c) \geq \int_0^1 \left[\left(\frac{\rho(c)}{|\sup(c_t^\perp)|} \right)^{\frac{1}{2}} \int_{S^1} |c_t^\perp| |c_\theta| d\theta \right] dt$$

which suggests that while traversing a given area, one should try to minimize $\rho(c)/|\sup(c_t^\perp)|$. The key point is that we can increase $|\sup(c_t^\perp)|$ indefinitely by replacing the part of the curve supporting c_t^\perp by a saw-tooth shaped curve of high

frequency and small amplitude. When $\rho = \ell$ and $|\sup(c_t^\perp)| < \ell$, we can increase $|\sup(c_t^\perp)|$ so that $\ell/|\sup(c_t^\perp)|$ tends to 1. When $\rho = e^{A\ell}$ and $|\sup(c_t^\perp)| < 1/A$, we can force $e^{A|\sup(c_t^\perp)|}/|\sup(c_t^\perp)|$ to equal its unique minimum Ae by making $|\sup(c_t^\perp)|$ equal $1/A$. (In the case of the metric G^o , $\rho = 1$ so that $\rho/|\sup(c_t^\perp)|$ tends to 0.) In order to obtain an upper bound for a general path, we break it up into a series of tiny bumps. When $\rho = \ell$, this method gives an upper bound for $\text{dist}_{G^\rho}(C, C')$ which coincides with the lower bound. When $\rho = e^{A\ell}$, the larger the value of $|\sup(c_t^\perp)|$, the greater the divergence between the upper bound obtained by this method and the lower bound since it is more efficient to create a large bump all at once instead a series of tiny bumps.

Rectangular Bumps

Let $c_o : S^1 \rightarrow \mathbb{R}^2$ be a smooth and free immersion. Let C_o be the corresponding curve in \mathbb{R}^2 . Let c_o be parametrized by the arclength so that θ parametrizes the scaled circle $S_{\ell_o}^1$ where ℓ_o is the length of C_o . For any function $u(\theta)$, let u' denote $du/d\theta$. Let n_o denote the normal vector ic'_o . Let κ_o denote the curvature of c_o .

Fix small positive numbers δ and ϵ such that $\delta < \ell_o$ and $\epsilon \|\kappa_o\|_{\infty, [0, \delta]} \ll 1$. Construct a "rectangular" bump, over C_o as follows:

$$c_1(\theta) = \begin{cases} c_o(\theta) + \epsilon n_o & \text{if } 0 < \theta < \delta \\ \{c_o(\theta) + sn_o | 0 \leq s \leq \epsilon\} & \text{if } \theta = 0, \delta \\ c_o(\theta) & \text{otherwise} \end{cases}$$

Let C_1 be the corresponding curve in \mathbb{R}^2 .

Lemma 3.3. *For a rectangular bump C_1 over a curve C_o , we have the following estimates:*

(i) If $\rho = \ell$,

$$\text{dist}_{G^\rho}(C_o, C_1) \leq \left[\frac{1 + \epsilon \|\kappa_o\|_{\infty, [0, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [0, \delta]}} \right]^2 (\text{area of the bump})$$

(ii) If $\rho = e^{A\ell}$ and $\delta < 1/A$,

$$\begin{aligned} & \text{dist}_{G^\rho}(C_o, C_1) \\ & \leq \left[\frac{1 + \epsilon \|\kappa_o\|_{\infty, [0, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [0, \delta]}} \right]^{3/2} e^{A(\ell_o + 2\epsilon - \delta)/2} \sqrt{Ae \frac{1 + \epsilon \|\kappa_o\|_{\infty, [0, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [0, \delta]}}} (\text{bump area}) \end{aligned}$$

Proof. We prove the lemma using a modification of the "teeth" construction of Michor and Mumford [2]. If $\rho = \ell$, choose $A < 1/\delta$. Approximate C_1 by a "trapezoidal" bump \tilde{C}_1 as follows. Replace C_o in the interval $[0, \delta]$ by a saw-tooth curve of height η and period $\frac{1}{m}$ such that its length equals $\frac{1}{A}$. This is done by growing teeth on C_o in time η . Move the saw-tooth curve at unit speed along the normals n_o keeping its end-points fixed, until it touches the upper edge of the bump. Finally, retract the teeth in time η . Formally, define a path $c(t, \theta) = c_o(\theta) + f(t, \theta)n_o$ where $f(t, \theta)$ is defined as follows.

$$f(t, \theta) = 0, \quad 0 \leq t \leq \epsilon \text{ and } \delta \leq \theta \leq \ell_o$$

For $0 \leq t \leq \eta$, $0 \leq k \leq m - 1$,

$$f(t, \theta) = \begin{cases} t \left(\frac{2m\theta}{\delta} - 2k \right) & \frac{2k}{2m} \leq \frac{\theta}{\delta} \leq \frac{2k+1}{2m} \\ t \left(2k + 2 - \frac{2m\theta}{\delta} \right) & \frac{2k+1}{2m} \leq \frac{\theta}{\delta} \leq \frac{2k+2}{2m} \end{cases}$$

For $\eta \leq t \leq \epsilon - \eta$,

$$f(t, \theta) = \begin{cases} \frac{\epsilon(t-\eta) + \eta(\epsilon-\eta-t)}{\epsilon-2\eta} \cdot \frac{2m\theta}{\delta} & 0 \leq \frac{\theta}{\delta} \leq \frac{1}{2m} \\ \frac{\epsilon-\eta}{\epsilon-2\eta}(t-\eta) + f(\eta, \theta) & \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1 - \frac{1}{2m} \\ \frac{\epsilon(t-\eta) + \eta(\epsilon-\eta-t)}{\epsilon-2\eta} \cdot 2m(1 - \frac{\theta}{\delta}) & 1 - \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1 \end{cases}$$

For $\epsilon - \eta \leq t \leq \epsilon$,

$$f(t, \theta) = \begin{cases} \frac{2m\epsilon\theta}{\delta} & 0 \leq \frac{\theta}{\delta} \leq \frac{1}{2m} \\ \frac{\epsilon[t-(\epsilon-\eta)] + (\epsilon-t)f(\epsilon-\eta, \theta)}{2m\epsilon(1 - \frac{\theta}{\delta})} & \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1 - \frac{1}{2m} \\ \frac{\eta}{2m\epsilon(1 - \frac{\theta}{\delta})} & 1 - \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1 \end{cases}$$

$$\begin{aligned} c' &= c'_o + f'n_o - f\kappa_o c'_o = (1 - f\kappa_o)c'_o + f'n_o \\ |c'| &= \sqrt{(1 - f\kappa_o)^2 + f'^2} \\ n &= \frac{-f'c'_o + (1 - f\kappa_o)n_o}{|c'|} \end{aligned}$$

$$\begin{aligned} c_t &= f_t n_o \\ c_t^\perp \cdot c_t^\perp &= (c_t \cdot n)^2 = \frac{(1 - f\kappa_o)^2 f_t^2}{|c'|^2} \end{aligned}$$

Let

$$\begin{aligned} \beta &= \sqrt{\frac{(1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2}{(1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2}} \\ 1 \leq \beta &\leq \frac{1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]}} \end{aligned}$$

Choose m and η such that $\int_o^\delta |c'(\eta, \theta)| d\theta = \frac{1}{A}$. Note that as $m \rightarrow \infty$, $\eta \rightarrow 0$. Estimates when $0 \leq t \leq \eta$: Since $|f'|$ is independent of θ and $|f| \leq \epsilon$,

$$\begin{aligned} \sqrt{(1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} &\leq |c'(\eta, \theta)| \leq \sqrt{(1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} \\ \delta \sqrt{(1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} &\leq \int_o^\delta |c'(\eta, \theta)| d\theta = \frac{1}{A} \leq \delta \sqrt{(1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} \end{aligned}$$

We also have

$$\begin{aligned} \sqrt{(1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} &\leq |c'(t, \theta)| \leq \sqrt{(1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} \\ \frac{1}{\beta} \sqrt{(1 + \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} &\leq |c'(t, \theta)| \leq \beta \sqrt{(1 - \epsilon\|\kappa_o\|_{\infty, [o, \delta]})^2 + f'^2} \\ \frac{1}{\beta} \cdot \frac{1}{A\delta} &\leq |c'(t, \theta)| \leq \frac{\beta}{A\delta} \end{aligned}$$

$$\begin{aligned} \ell(c) &= \int_o^{\ell_o} |c'| d\theta \leq (\ell_o - \delta) + \frac{\beta}{A} \\ e^{A\ell(c)} &\leq e^{A(\ell_o - \delta)} e^\beta \end{aligned}$$

Since $|f_t| \leq 1$,

$$\int_o^{\ell_o} |c_t^\perp|^2 |c'| d\theta = \int_o^\delta \frac{(1 - f\kappa_o)^2 f_t^2}{|c'|} d\theta \leq (1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]})^2 \beta A \delta^2$$

$$\lim_{m \rightarrow \infty} \int_o^\eta \left[\rho(\ell(c)) \int_o^{\ell_o} |c_t^\perp|^2 |c'| d\theta \right]^{1/2} dt = 0$$

Estimates when $\eta \leq t \leq \epsilon - \eta$: Estimate for $|c'(t, \theta)|$ is the same as in the interval $[\frac{\delta}{2m}, \delta(1 - \frac{1}{2m})]$ since the curve has the same shape. In the intervals $[0, \frac{\delta}{2m}]$ and $[\delta(1 - \frac{1}{2m}), \delta]$, $\frac{2m\eta}{\delta} \leq |f'(t, \theta)| \leq \frac{2m\epsilon}{\delta}$. Therefore, $|c'(\eta, \theta)| \leq |c'(t, \theta)| \leq 1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]} + \frac{2m\epsilon}{\delta}$.

$$\ell(c) = \int_o^{\ell_o} |c'| d\theta \leq (\ell_o - \delta) + \left(1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]} + \frac{2m\epsilon}{\delta}\right) \frac{\delta}{m} + \frac{\beta}{A}$$

$$\lim_{m \rightarrow \infty} \ell(c) \leq (\ell_o - \delta) + 2\epsilon + \frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}} \frac{1}{A}$$

$$\lim_{m \rightarrow \infty} e^{A\ell(c)} \leq e^{A\ell_o + 2\epsilon - \delta} e^{\frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}$$

We also have $|f_t| \leq \frac{\epsilon - \eta}{\epsilon - 2\eta}$. Therefore,

$$\int_o^{\ell_o} |c_t^\perp|^2 |c'| d\theta \leq (1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]})^2 \left(\frac{\epsilon - \eta}{\epsilon - 2\eta}\right)^2 \beta A \delta^2$$

$$\lim_{m \rightarrow \infty} \int_\eta^{\epsilon - \eta} \left[\rho(\ell(c)) \int_o^{\ell_o} |c_t^\perp|^2 |c'| d\theta \right]^{1/2} dt$$

$$\leq \begin{cases} (1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}) \sqrt{\frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}} \left[A(\ell_o + 2\epsilon - \delta) + \frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}} \right]^{1/2} \delta \epsilon & \text{if } \rho(\ell) = \ell \\ (1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}) \sqrt{\frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}} e^{A(\ell_o + 2\epsilon)/2} \sqrt{A e^{\frac{1 + \epsilon \|\kappa_o\|_{\infty, [o, \delta]}}{1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}} - A\delta}} \delta \epsilon & \text{if } \rho(\ell) = e^{A\ell} \end{cases}$$

Estimates when $\epsilon - \eta \leq t \leq \epsilon$: The path $c(t, \theta)$ in the interval $[\epsilon - \eta, \epsilon]$ is essentially the same as that in $[0, \eta]$ and

$$\lim_{m \rightarrow \infty} \int_{\epsilon - \eta}^\epsilon \left[\rho(\ell(c)) \int_o^{\ell_o} |c_t^\perp|^2 |c'| d\theta \right]^{1/2} dt = 0$$

Since

$$(\text{area of the bump}) = \int_o^\epsilon \int_o^\delta (1 - t\kappa_o) d\theta dt \geq (1 - \epsilon \|\kappa_o\|_{\infty, [o, \delta]}) \delta \epsilon$$

the lemma is proved in the case when $\rho = e^{A\ell}$. It is proved in the case when $\rho = \ell$ by letting $A \rightarrow 0$. \square

In Lemma 3.3, we may replace the single bump by a finite number of disjoint bumps of height ϵ and total length δ . The proof remains unchanged except that we

must replace 2ϵ in the formula by $2k\epsilon$ if k is the number of bumps. The function f in each individual bump may be positive or negative.

We prove the theorem by approximating the path by a series of small rectangular bumps. The error of approximation may be made arbitrarily small by the following lemma. Define a *Fréchet* metric on $\text{Imm}(S^1, \mathbb{R}^2)$ and $B_i(S^1, \mathbb{R}^2)$ as follows. If c_o, c_1 are points in $\text{Imm}(S^1, \mathbb{R}^2)$, let

$$d_\infty(c_o, c_1) = \sup_\theta |c_o(\theta) - c_1(\theta)|$$

If $C_o, C_1 \in B_i(S^1, \mathbb{R}^2)$, let

$$d_\infty(C_1, C_2) = \inf_{\{c_o, c_1 | \pi(c_o) = C_o, \pi(c_1) = C_1\}} d_\infty(c_o, c_1)$$

Lemma 3.4. *For any pair $C_1, C_2 \in B_i(S^1, \mathbb{R}^2)$,*

$$\text{dist}_{G^\rho}(C_1, C_2) \leq d_\infty(C_1, C_2) \cdot \max\{\rho(\ell_1), \rho(\ell_2)\}$$

where $\ell_i = \ell(C_i), i = 1, 2$.

Proof. Let c_1, c_2 be lifts of C_1, C_2 to $\text{Imm}(S^1, \mathbb{R}^2)$. Let $c(t, \theta) = (1-t)c_1(\theta) + tc_2(\theta)$ be a path connecting them. Then, $|c_\theta(t)| \leq (1-t)|c_{1,\theta}| + t|c_{2,\theta}|$ and hence, $\ell(c(t)) \leq \max\{\ell(C_1), \ell(C_2)\}$. Moreover, $c_t = c_2 - c_1$. Therefore,

$$\begin{aligned} \text{dist}_{G^\rho}(C_1, C_2) &\leq \inf_c L_{G^\rho}(c) \\ &\leq \inf_{\{pairs\ c_1, c_2\}} \left\{ \sup_\theta |c_1(\theta) - c_2(\theta)| \right\} \max\{\rho(\ell_1), \rho(\ell_2)\} \end{aligned}$$

□

The polygonal approximations used in the proof of the theorem lie on the boundary of $\text{Imm}(S^1, \mathbb{R}^2)$ and $B_i(S^1, \mathbb{R}^2)$, and Lemma 3.4 extends to them.

Proof of the theorem:

Consider a path $c(t, \theta)$ connecting C and C' . Since the absolute curvature of the curves $c(t, \cdot)$ is uniformly bounded by a constant K , each curve has a tubular neighborhood of width which is bounded from below. Choose ϵ and a sequence

$$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$$

such that $\epsilon K \ll 1$ and, for $0 \leq k < N$, $c(t_{k+1}, \cdot)$ is in a local chart of $c(t_k, \cdot)$:

$$c(t_{k+1}, \theta) = c(t_k, \theta) + f_k(\theta) n_k$$

where $c(t_k, \cdot)$ is parametrized by the arclength, n_k is the normal vector field of $c(t_k, \cdot)$ and $|f_k| < \epsilon$. Let $F = \max\{\|f'_k(\theta)\|_\infty | 0 \leq k < N\}$. Let C_k denote $\pi(c(t_k, \cdot))$. Let $\ell_k = \ell(C_k)$. Let $\tilde{\ell}_k = (1 + \epsilon K + F)\ell_k$.

Choose δ such that

$$\max_k \sqrt{\tilde{\ell}_k \rho(\tilde{\ell}_k)} \cdot F \delta < \frac{\epsilon}{N}$$

We now estimate the distances $\text{dist}_{G^\rho}(C_k, C_{k+1})$. Consider the path segment $[c_o, c_1]$ from c_o to c_1 in the local chart at c_o . Divide the range of θ into intervals of length δ . Replace f_o by a piecewise constant function \tilde{f}_o whose value in each subinterval equals the average of f_o over that interval. Let \tilde{C}_o be the curve defined by \tilde{f}_o . The *Fréchet* distance between C_1 and \tilde{C}_o is $\leq F\delta$. The sum of the jumps

in \bar{f}_o is $\leq F\ell_o$. Since $|\bar{c}'_o| = |1 - \epsilon\kappa_o| \leq 1 + \epsilon K$, $\ell(\bar{C}_o) \leq (1 + \epsilon K + F)\ell_o = \tilde{\ell}_o$. Therefore,

$$\text{dist}_{G^\rho}(C_1, \bar{C}_o) \leq \max\{\tilde{\ell}_o\rho(\tilde{\ell}_o), \tilde{\ell}_1\rho(\tilde{\ell}_1)\} \cdot F\delta \leq \frac{\epsilon}{N}$$

Let $\alpha([c_o, c_1])$ denote the area swept out by the path c during $[0, t_1]$. The area between C_o and \bar{C}_o equals the area between C_o and C_1 which in turn is less than or equal to $\alpha([c_o, c_1])$. The curve \bar{C}_o consists of a series of bumps over C_o . Traverse the bumps sequentially, taking care to retract the common edge of each bump with the previous bump before going to the next bump. Retracting a common edge can be done without incurring any cost.

By Lemma 3.3, if $\rho = \ell$,

$$\text{dist}_{G^\rho}(C_o, \bar{C}_o) \leq \left(\frac{1 + \epsilon K}{1 - \epsilon K}\right)^2 \alpha([c_o, c_1])$$

and if $\rho = e^{A\ell}$,

$$\text{dist}_{G^\rho}(C_o, \bar{C}_o) \leq \left(\frac{1 + \epsilon K}{1 - \epsilon K}\right)^{3/2} e^{A(\ell_o + 2\epsilon)/2} \sqrt{Ae^{\frac{1+\epsilon K}{1-\epsilon K}}} \alpha([c_o, c_1])$$

$$\text{dist}_{G^\rho}(C_o, C_1) \leq \text{dist}_{G^\rho}(C_o, \bar{C}_o) + \frac{\epsilon}{N}$$

Similar estimates hold for $\text{dist}_{G^\rho}(C_k, C_{k+1})$ for $0 < k < N$.

Therefore, if $\rho = \ell$,

$$\text{dist}_{G^\rho}(C, C') \leq \left(\frac{1 + \epsilon K}{1 - \epsilon K}\right)^2 \alpha(c) + \epsilon$$

and if $\rho = e^{A\ell}$,

$$\text{dist}_{G^\rho}(C, C') \leq \left(\frac{1 + \epsilon K}{1 - \epsilon K}\right)^{3/2} e^{A\ell_{\max}(c)/2} \sqrt{Ae^{\frac{1+\epsilon K}{1-\epsilon K}}} \alpha(c) + \epsilon$$

Since ϵ is arbitrary, we have

$$\text{dist}_{G^\rho}(C, E) \leq \begin{cases} \alpha(c) & \text{if } \rho(\ell) = \ell \\ e^{A\ell_{\max}(c)/2} \sqrt{Ae^{\frac{1+\epsilon K}{1-\epsilon K}}} \alpha(c) & \text{if } \rho(\ell) = e^{A\ell} \end{cases}$$

□

For any *oriented curve* C^{or} , define the integer-valued measurable function w_C on \mathbb{R}^2 by:

$$w_C(x, y) = \text{the winding number of } C \text{ around } (x, y)$$

and let

$$d^b(C_1^{or}, C_2^{or}) = \int_{\mathbb{R}^2} |w_{C_1} - w_{C_2}| dx dy$$

It is shown in [2] that for any two *oriented curves* C_1^{or}, C_2^{or} ,

$$d^b(C_1^{or}, C_2^{or}) \leq \min_{\text{all paths } c \text{ joining } C_1, C_2} \alpha(c)$$

Therefore, we have

$$\text{dist}_{G^\rho}(C_1^{or}, C_2^{or}) \geq \begin{cases} d^b(C_1^{or}, C_2^{or}) & \text{if } \rho(\ell) = \ell \\ \sqrt{Ae^{\frac{1+\epsilon K}{1-\epsilon K}}} d^b(C_1^{or}, C_2^{or}) & \text{if } \rho(\ell) = e^{A\ell} \end{cases}$$

Corollary 3.5. (Existence of minimal geodesics) If $\rho = \ell$, then the only minimal geodesics are the horizontal paths along which $|c_t^\perp| \ell$ is constant.

Proof. If $c(t)$ is horizontal path projecting to a minimal geodesic, $L_{G^\rho}(c) = \alpha(c)$. Hence, since the inequality

$$L_{G^\rho}(c) \geq \int_0^1 \left[\int_{S^1} |c_t^\perp| |c_\theta| d\theta \right] dt = \alpha(c)$$

is an equality if and only if $|c_t^\perp|$ does not depend on θ , that is, $\frac{\partial}{\partial \theta} |c_t^\perp| = 0$ (the case of "grassfire"), $|c_t^\perp|$ must be independent of θ . Let $c(t, \theta)$ be a horizontal path connecting C_1, C_2 such that $|c_t^\perp|$ is independent of θ . After reparametrization if necessary, we may assume that $c_t \cdot c_\theta = 0$. Following [2], we let $w(c)$ be the 2-current defined by the path $c(t, \theta)$. Since $c(t, \theta)$ is an immersion,

$$d^\flat(C_1^{or}, C_2^{or}) = \int_{\mathbb{R}^2} |w(c)| dx dy = \int_{S^1 \times [0,1]} |\det dc(t, \theta)| d\theta dt = \alpha(c)$$

Therefore, $L_{G^\rho}(c)$ is the minimal distance between C_1, C_2 . For $c(t, \theta)$ to be a geodesic path, reparametrize t such that the infinitesimal arc-length $|c_t^\perp| \ell$ is constant along the path. \square

Corollary 3.6. Suppose $\rho = e^{A\ell}$ and $c(t, \theta), 0 \leq t \leq 1$, is a path connecting C_1, C_2 . Assume that $|c_t^\perp|$ does not depend on θ and $\ell(t) < 1/A$ for all t . Then,

$$\text{dist}_{G^\rho}(C_1^{or}, C_2^{or}) = \sqrt{Ae} d^\flat(C_1^{or}, C_2^{or})$$

It follows that the path $c(t, \theta)$ is not minimal.

Proof. Break up the interval $[0, 1]$ into small segments of length ϵ and apply Lemma 3.3 with $\delta = \ell_\epsilon$. (The proof of the lemma extends to this case after minor modifications.) Calculate the length of the new path \tilde{c}_ϵ applying the construction of Lemma 3.3 to each of the segments.. We get $\lim_{\epsilon \rightarrow 0} L_{G^\rho}(\tilde{c}_\epsilon) = \sqrt{Ae} \alpha(c)$ as in the proof of Theorem 3.1. On the other hand,

$$\begin{aligned} L_{G^\rho}(c) &= \int_0^1 \sqrt{\frac{\rho(c)}{\ell(c)}} \left[\int_{S^1} |c_t^\perp| |c_\theta| d\theta \right] dt \\ &> \sqrt{\min \frac{\rho}{\ell}} \alpha(c) = \sqrt{Ae} \alpha(c) \end{aligned}$$

\square

4. GEODESIC EQUATION

With $\psi = \frac{1}{2} \log \rho$, we have the following relation between the Levi-Civita connections [1]:

$$\nabla_X^\rho Y = \nabla_X^o Y + (D_X \psi) Y + (D_Y \psi) X - G^o(X, Y) \nabla^o \psi$$

Let $c(t)$ be a horizontal path in $\text{Imm}(S^1, \mathbb{R}^2)$ projecting to a path in $B_i(S^1, \mathbb{R}^2)$. c_t is a horizontal vector field along c projecting onto $B_i(S^1, \mathbb{R}^2)$. Write $c_t = a n_c$. Then the geodesic curvature $\nabla_{c_t}^o c_t$ is horizontal and is given by the formula [2]

$$\nabla_{c_t}^o c_t = \left(a_t - \frac{1}{2} \kappa_c a^2 \right) n_c$$

Therefore,

$$\nabla_{c_t}^\rho c_t = \left(a_t - \frac{1}{2} \kappa_c a^2 + 2\psi_t a - \|a n_c\|^2 (\nabla^o \psi \cdot n_c) \right) n_c$$

The geodesic equation may be written as

$$a_t = \frac{1}{2}\kappa_c a^2 - 2\psi_t a + \|an_c\|_o^2 (\nabla^o \psi \cdot n_c)$$

Suppose ρ is a function of ℓ alone: $\rho = \rho(\ell)$. Let ρ' denote $d\rho/d\ell$. Let $\overline{fg} = \frac{1}{\ell} \int_{S^1} fg|c_\theta|d\theta$. Then,

$$\begin{aligned} \psi_t &= \frac{\rho'}{2\rho} \ell_t = -\frac{\rho' \ell}{2\rho} \overline{a\kappa_c} \\ \nabla^o \psi \cdot n_c &= \frac{\rho'}{2\rho} (\nabla^o \ell \cdot n_c) = -\frac{\rho'}{2\rho} \kappa_c \end{aligned}$$

Therefore the geodesic equation in the metric $G^{\rho(\ell)}$ is

$$a_t = \frac{\kappa_c}{2} \left(a^2 - \frac{\rho' \ell}{\rho} \overline{a^2} \right) + a \frac{\rho' \ell}{\rho} \overline{a\kappa_c}$$

Specifically,

$$(4.1) \quad a_t = \begin{cases} \frac{\kappa_c}{2} \left(a^2 - \overline{a^2} \right) + a \cdot \overline{a\kappa_c} & \text{if } \rho(\ell) = \ell \\ \frac{\kappa_c}{2} \left(a^2 - (A\ell) \overline{a^2} \right) + (A\ell) a \cdot \overline{a\kappa_c} & \text{if } \rho(\ell) = e^{A\ell} \end{cases}$$

As an example, consider the case of concentric circles, $c(t, \theta) = r(t)e^{i\theta}$, $r_o = r(0)$, $r_1 = r(1)$. We have $\kappa_c = 1/r$ and $a = -r_t$. Substituting these in Eq. (1) when $\rho(\ell) = \ell$, we get $-r_{tt} = r_t^2/r$ or $(r^2)_{tt} = 0$. Therefore, $r^2(t) = tr_1^2 + (1-t)r_o^2$. This example is a special case of the curve evolution by "grassfire" in which a is independent of θ . We have $a^2 = \overline{a^2}$ and the equation of the geodesic reduces to $a_t = a \cdot \overline{a\kappa_c} = -a\ell_t/\ell$ and hence $(a\ell)_t = 0$. Therefore, $a\ell = \text{a constant}$. By substituting in the equation for the length of the geodesic, we find that $a\ell = \text{the length of the geodesic}$. When $\rho(\ell) = e^{A\ell}$, the equation of the geodesic in the case of concentric circles is $(r^2)_{tt} = r_t^2(1 - 2\pi r A)$ which is zero when the perimeter of the circle equals $1/A$, marking the unique inflection point of the function $r^2(t)$.

5. SECTIONAL CURVATURE

We will use the local chart on $B_i(S^1, \mathbb{R}^2)$ constructed in [2]. Let $c \in \text{Imm}(S^1, \mathbb{R}^2)$ be a smooth free immersion, $c : S^1 \rightarrow \mathbb{R}^2$. Let $C = \pi(c) \in B_i(S^1, \mathbb{R}^2)$. Let c be parametrized by the arclength so that θ parametrizes the scaled circle S_ℓ^1 where ℓ is the length of c . A local chart centered at C is as follows:

$$\begin{aligned} \Xi : C^\infty(S_\ell^1, (-\epsilon, \epsilon)) &\rightarrow \text{Imm}(S^1, \mathbb{R}^2) \\ \Xi(f)(\theta) &= c(\theta) + f(\theta)n_c(\theta) \\ \pi \circ \Xi : C^\infty(S_\ell^1, (-\epsilon, \epsilon)) &\rightarrow B_{i,f}(S^1, \mathbb{R}^2) \end{aligned}$$

For $h \in C^\infty(S_\ell^1, \mathbb{R})$, $h \cdot n_c \in T_{\Xi(f)} \text{Imm}(S^1, \mathbb{R}^2)$. If u is a function on S_ℓ^1 , let u' denote its derivative $du/d\theta$; . We have the following formulae from [2]:

$$\begin{aligned} \Xi(f)' &= (1 - f\kappa_c)c' + f'n_c \\ |\Xi(f)'| &= 1 - f\kappa_c + \frac{1}{2}f'^2 + O(f^3) \\ \ell_f &= \int_{S_\ell^1} \sqrt{(1 - f\kappa_c)^2 + f'^2} d\theta = \int_{S_\ell^1} \left(1 - f\kappa_c + \frac{1}{2}f'^2 \right) d\theta + O(f^3) \end{aligned}$$

$$\kappa_f = \kappa_c + (f'' + f\kappa_c^2) + (f^2\kappa_c^3 + \frac{1}{2}f'^2\kappa_c + ff'\kappa_c' + 2ff''\kappa_c) + O(f^3)$$

$$R_C^o(m, h, m, h) = \frac{1}{2} \|(m'h - mh')\|^2$$

The conformal change in the Riemann (4,0) curvature tensor is given by the formula [1]

$$(5.1) \quad \begin{aligned} R^\rho &= \rho(R^o - G^o \otimes \Omega) \\ \Omega &= (\nabla^o d\psi - d\psi \circ d\psi + \frac{1}{2}\|d\psi\|^2 G^o) \end{aligned}$$

where \otimes denotes the Kulkarni-Nomizu product of symmetric 2-tensors:

$$u \otimes v(x, y, z, t) = u(x, z)v(y, t) + u(y, t)v(x, z) - u(x, t)v(y, z) - u(y, z)v(x, t)$$

$\nabla^o d\psi$ is the Hessian of ψ with respect to G^o and \circ denotes the symmetric product of symmetric tensors:

$$d\psi \circ d\psi(x, y) = d\psi(x)d\psi(y)$$

Let $m, h \in C^\infty(S_\ell^1, \mathbb{R})$ be constant tangent vectors in the local chart which are orthonormal with respect to G^ρ . Let $\langle u, v \rangle$ denote $G^o(u, v)$. Note that $\|m\|^2 = \|h\|^2 = 1/\rho$ and $\langle m, h \rangle = 0$. From (5.1), we obtain a formula for the sectional curvature:

$$K_{B_i, C}^\rho(m, h) = \frac{\rho}{2} \|(m'h - mh')\|^2 - \Omega_c(m, m) - \Omega_c(h, h)$$

An explicit expression for the Hessian is

$$(\nabla^o d\psi)_c(m, m) = D_{c, m}(D_{c, m}\psi) - D_{\Gamma^o(m, m)}\psi$$

where the Christoffel symbol $\Gamma^o(m, m) = -\frac{1}{2}\kappa_c m^2$ as shown in [2]. (Note that the sign convention in this paper is opposite to that used in [2].) Therefore,

$$\Omega_c(m, m) = D_{c, m}(D_{c, m}\psi) + \langle \frac{1}{2}\kappa_c m^2, \nabla^o \psi \rangle - \langle m, \nabla^o \psi \rangle^2 + \frac{1}{2\rho} \|\nabla^o \psi\|^2$$

Since

$$\nabla^o \psi = \frac{1}{2\rho} \nabla^o \rho$$

and

$$D_{c, m}(D_{c, m}\psi) = \frac{1}{2\rho} D_{c, m}(D_{c, m}\rho) - \frac{1}{2\rho^2} \langle m, \nabla^o \psi \rangle^2$$

we have

$$(5.2) \quad \begin{aligned} \Omega_c(m, m) &= \frac{1}{2\rho} D_{c, m}(D_{c, m}\rho) + \frac{1}{4\rho} \langle \kappa_c m^2, \nabla^o \rho \rangle - \frac{3}{4\rho^2} \langle m, \nabla^o \rho \rangle^2 \\ &+ \frac{1}{8\rho^3} \|\nabla^o \rho\|^2 \end{aligned}$$

5.1. **Case when ρ is a function of ℓ alone:** $\rho = \rho(\ell)$. We compute the various quantities involved in the expression for the sectional curvature using the local chart.

$$\begin{aligned} D_{c,m}\ell &= -\langle m, \kappa_c \rangle \\ \nabla^o \rho &= -\rho' \kappa_c \end{aligned}$$

$$\begin{aligned} D_{c,m}(D_{c,m}\rho) &= \rho' D_{c,m}(D_{c,m}\ell) + \rho'' (D_{c,m}\ell)^2 \\ &= \rho' \|m'\|^2 + \rho'' \langle m, \kappa_c \rangle^2 \end{aligned}$$

$$\begin{aligned} \langle \kappa_c m^2, \nabla^o \rho \rangle &= -\rho' \|m\kappa_c\|^2 \\ \langle m, \nabla^o \rho \rangle^2 &= \rho'^2 \langle m, \kappa_c \rangle^2 \end{aligned}$$

$$\|\nabla^o \rho\|^2 = \rho'^2 \|\kappa_c\|^2$$

$$\Omega_c(m, m) = \frac{\rho'}{2\rho} \|m'\|^2 - \frac{\rho'}{4\rho} \|m\kappa_c\|^2 - \frac{3\rho'^2 - 2\rho\rho''}{4\rho^2} \langle m, \kappa_c \rangle^2 + \frac{\rho'^2}{8\rho^3} \|\kappa_c\|^2$$

Substituting the explicit expressions in the formula for the sectional curvature, we get

$$\begin{aligned} (5.3) \quad K_{B_i, C}^{\rho(\ell)}(m, h) &= \frac{\rho}{2} \|m'h - mh'\|^2 - \frac{\rho'}{2\rho} (\|m'\|^2 + \|h'\|^2) \\ &+ \frac{\rho'}{4\rho} (\|m\kappa_c\|^2 + \|h\kappa_c\|^2) \\ &+ \frac{3\rho'^2 - 2\rho\rho''}{4\rho^2} (\langle m, \kappa_c \rangle^2 + \langle h, \kappa_c \rangle^2) - \frac{\rho'^2}{4\rho^3} \|\kappa_c\|^2 \end{aligned}$$

Each of the last three terms on the right-hand side is absolutely bounded with respect to m and h since:

$$\|m\kappa_c\|^2 + \|h\kappa_c\|^2 \leq \frac{2\|\kappa_c\|_\infty^2}{\rho}, \quad \langle m, \kappa_c \rangle^2 + \langle h, \kappa_c \rangle^2 \leq \frac{2\|\kappa_c\|^2}{\rho}$$

where $\|\kappa_c\|_\infty = \max_\theta |\kappa_c(\theta)|$. Therefore, the boundedness of the sectional curvature from above depends on the first two terms. For a fixed m , the magnitude of each of the two terms depends on $\|h'\|$ which can be made arbitrarily large while keeping $\|h\|$ fixed by making h highly oscillatory.

Proposition 5.1. Let m be a tangent vector at $C\epsilon B_i$. Then the sectional curvature $K_{B_i, C}^{\rho(\ell)}(m, h)$ is uniformly bounded from above with respect to h if and only if $m^2(\theta) \leq \frac{\rho'\ell}{\rho} m^2$.

Proof. We may assume that m, h are orthonormal with respect to the metric G^ρ so that $\frac{\rho'\ell}{\rho} m^2 = \frac{\rho'}{\rho^2}$. We need to estimate only the first two terms on the right-hand side of Eq. (5.3).

Suppose $\|m\|_\infty^2 \leq \frac{\rho'}{\rho^2}$. Then,

$$\begin{aligned}
 & \frac{\rho}{2} \|m'h - mh'\|^2 - \frac{\rho'}{2\rho} (\|m'\|^2 + \|h'\|^2) \\
 &= \rho \int_{S_\ell^1} \left[\frac{\rho}{2} (m'h)^2 - \frac{\rho'}{2\rho} m'^2 - \rho m m' h h' + \frac{\rho}{2} \left(m^2 - \frac{\rho'}{\rho^2} \right) h'^2 \right] d\theta \\
 &\leq \frac{\rho}{2} \|m'\|_\infty^2 - \frac{\rho^2}{4} \int_{S_\ell^1} (m^2)' (h^2)' d\theta \\
 &\leq \frac{\rho}{2} \|m'\|_\infty^2 + \frac{\rho^2}{4} \int_{S_\ell^1} (m^2)'' (h^2) d\theta \\
 &\leq \frac{\rho}{2} \|m'\|_\infty^2 + \frac{\rho}{4} \|(m^2)''\|_\infty < \infty
 \end{aligned}$$

Conversely, suppose $\|m\|_\infty^2 > \frac{\rho'}{\rho^2}$. Choose ϵ such that $U = \{\theta : m^2(\theta) > \rho'/\rho^2 + \epsilon\}$ is not empty. Let h be a high frequency wave function with $\text{supp}(h) \subset U$. Then,

$$\begin{aligned}
 & \frac{\rho}{2} \|m'h - mh'\|^2 - \frac{\rho'}{2\rho} (\|m'\|^2 + \|h'\|^2) \\
 &= \rho \int_{S_\ell^1} \left[\frac{\rho}{2} (m'h)^2 - \frac{\rho'}{2\rho} m'^2 - \rho m m' h h' + \frac{\rho}{2} \left(m^2 - \frac{\rho'}{\rho^2} \right) h'^2 \right] d\theta \\
 &\geq -\frac{\rho'}{2\rho} \|m'\|^2 + \frac{\rho^2}{4} \int_{S_\ell^1} (m^2)'' (h^2) d\theta + \frac{\rho\epsilon}{2} \|h'\|^2 \\
 &\geq -\frac{\rho'}{2\rho} \|m'\|^2 - \frac{\rho^2}{4} \|(m^2)''\|_\infty + \frac{\rho\epsilon}{2} \|h'\|^2
 \end{aligned}$$

which tends to ∞ as the frequency of the wave function h tends to ∞ . □

If $U = \{\theta : m^2(\theta) > \rho'/\rho^2 + \epsilon\}$ is not empty,

$$\begin{aligned}
 1 &= \rho \left[\int_U m^2 d\theta + \int_{[0,\ell]/U} m^2 d\theta \right] \\
 &\geq \rho \int_U m^2 d\theta \geq \left(\frac{\rho'}{\rho} + \epsilon\rho \right) |U|
 \end{aligned}$$

and hence, $|U| < \rho/\rho'$. If $\rho = \ell$, $|U| < \ell$ and if $\rho = e^{A\ell}$, $|U| < 1/A$. Thus, the case when $\|m\|_\infty^2 > \rho'/\rho^2$ may be seen as a generalization of the rectangular bump considered in §3.

If $\rho = \ell$, the sectional curvature is bounded if and only if $m^2(\theta) \leq \overline{m^2}$ which is true if and only if $m = 1/\ell$. Setting $m = 1/\ell$, we get

$$K_{B_i, C}^{\rho(\ell)}(m, h) = \frac{\|h\kappa_c\|^2}{4\ell} + \frac{3}{4\ell^2} \left(\langle \frac{1}{\ell}, \kappa_c \rangle^2 + \langle h, \kappa_c \rangle^2 \right)$$

which is always positive. If h and κ_c additionally have disjoint supports, the sectional curvature equals $3\pi^2 N^2/\ell^4$ where N is the rotation index of C .

If $\rho = e^{A\ell}$, the sectional curvature is bounded if and only if $m^2(\theta) \leq (A\ell)\overline{m^2}$. In particular, the sectional curvature is unbounded for every m if $\ell < 1/A$.

For an example of a negative sectional curvature, consider the unit square with slightly rounded corners. Choose m, h such that $\text{supp}(m)$ and $\text{supp}(h)$ are disjoint and concentrated along the straight portions of the square. Then,

$$K_{B_i, C}^{\rho(l)}(m, h) = -\frac{\rho'}{2\rho} \left(\|m'\|^2 + \|h'\|^2 \right) - \frac{\rho'^2}{4\rho^3} \|\kappa_c\|^2$$

5.2. **Case when** $\rho = \int_{S^1} \varphi(\kappa^2) |c_\theta| d\theta$. We again assume that the tangent vectors $m, h \in C^\infty(S_\ell^1, \mathbb{R})$ are constant in the local chart and are orthonormal with respect to G^ρ .

$$\begin{aligned} D_{c,m}|c_\theta| &= -\kappa_c m \\ D_{c,m}(D_{c,m}|c_\theta|) &= m'^2 \\ D_{c,m}\kappa &= m'' + \kappa_c^2 m \\ D_{c,m}(D_{c,m}\kappa) &= 2\kappa_c^3 m^2 + \kappa_c m'^2 + 2\kappa_c' m m' + 4\kappa_c m m'' \\ D_{c,m}\rho &= \int_{S^1} [2\kappa_c \varphi'(m'' + \kappa_c^2 m) - \varphi \kappa_c m] |c_\theta| d\theta \\ &= \int_{S^1} [(2\kappa_c \varphi')'' + 2\kappa_c^3 \varphi' - \kappa_c \varphi] m |c_\theta| d\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla^o \rho &= (2\kappa_c \varphi')'' + 2\kappa_c^3 \varphi' - \kappa_c \varphi \\ D_{c,m}(D_{c,m}\rho) &= \int_{S^1} [2\kappa_c \varphi' (4\kappa_c m m'' + 2\kappa_c' m m' + \kappa_c m'^2 + 2\kappa_c^3 m^2) \\ &\quad + 2(2\kappa_c^2 \varphi'' + \varphi')(m'' + \kappa_c^2 m)^2 \\ &\quad - 4\kappa_c^2 \varphi' m (m'' + \kappa_c^2 m) + \varphi m'^2] |c_\theta| d\theta \\ &= \int_{S^1} [2(2\kappa_c^2 \varphi'' + \varphi') m''^2 + 8\kappa_c^2 (\kappa_c^2 \varphi'' + \varphi') m m'' \\ &\quad + (2\kappa_c^2 \varphi' + \varphi) m'^2 + 4\kappa_c \kappa_c' \varphi' m m' \\ &\quad + 2\kappa_c^4 (2\kappa_c^2 \varphi'' + \varphi') m^2] |c_\theta| d\theta \end{aligned}$$

Applying integration by parts to the second term, we get

$$\begin{aligned} D_{c,m}(D_{c,m}\rho) &= \int_{S^1} [2(2\kappa_c^2 \varphi'' + \varphi') m''^2 - (8\kappa_c^4 \varphi'' + 6\kappa_c^2 \varphi' - \varphi) m'^2 \\ &\quad - 4\kappa_c (2\kappa_c^3 \varphi''' + 2\kappa_c (4\kappa_c \kappa_c' + 1) \varphi'' + 3\kappa_c' \varphi') m m' \\ &\quad + 2\kappa_c^4 (2\kappa_c^2 \varphi'' + \varphi') m^2] |c_\theta| d\theta \end{aligned}$$

Since there exists a constant γ such that for all $f \in C^2(S^1)$, if $0 < \epsilon \leq 1$,

$$\|u'\|^2 \leq \gamma(\epsilon \|u''\|^2 + \frac{1}{\epsilon} \|u\|^2)$$

boundedness of $K_{B_i, C}^\rho(m, h)$ from above at C for a given m is clearly controlled by the term

$$-\frac{1}{\rho} \int_{S^1} (2\kappa_c^2 \varphi'' + \varphi') h''^2 |c_\theta| d\theta$$

which is negative if $2x\varphi''(x) + \varphi'(x) > 0$. If $x > 0$,

$$\frac{1}{2\sqrt{x}} (2x\varphi''(x) + \varphi'(x)) = \sqrt{x}\varphi''(x) + \frac{\varphi'(x)}{2\sqrt{x}} = (\sqrt{x}\varphi'(x))'$$

Therefore, for a fixed m , $K_{B_i, C}^\rho(m, h)$ is bounded from above at C if $\varphi'(0) > 0$ and $\sqrt{x}\varphi'(x)$ is a strictly increasing function. A case of particular interest in Computer Vision is $\varphi(x) = (1 + Ax)^\alpha$ with $\alpha, A > 0$. We then have

$$2x\varphi''(x) + \varphi'(x) = [A\alpha(1 + Ax)^{\alpha-2}][Ax(2\alpha - 1) + 1]$$

which is positive if $\alpha \geq 1/2$.

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