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Curve Evolution and Segmentation Functionals: Application to Color Images

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Abstract[1]

In recent years, curve evolution has developed into an important tool in Computer Vision and has been applied to a wide variety of problems such as smoothing of shapes, shape analysis and shape recovery. The different versions of curve evolution used in Computer Vision together with the preprocessing step of constructing an edge-strength function can be integrated in the form of a new segmentation functional. The new functional permits junctions such as triple points to develop. The numerical solutions obtained retain sharp discontinuities or "shocks", thus providing sharp demarcation of object boundaries. In this paper, the new segmentation functional is extended for application to vector-valued features such as color.

1. Introduction

In recent years, curve evolution has been applied to a wide variety of problems such as smoothing of shapes [5,12], shape analysis [5,6] and shape recovery [3,4,8,9,14,15,19]. The underlying principle is the evolution of a simple closed curve whose points move in the direction of the normal with a prescribed velocity. Kimia, Tannenbaum and Zucker [5] proposed evolution of the curve by letting its points move with velocity consisting of two components: a component proportional to curvature and a constant component corresponding to morphology. The formulation involves one parameter which together with time provides a two-dimensional scale space, called "entropy" scale space of the shape. Keeping track of how singularities develop and disappear as the curve evolves provides information regarding the geometry of the shape in terms of its parts and its skeleton [6,7,17]. The above technique assumes that the object boundary (in the form of a simple closed curve) has already been extracted. As a result, attempts have been made in the last couple of years to extend the technique to recover shapes from noisy images in two distinctly different ways. In both cases, a continuous edge strength function v, varying between 0 and 1, is defined over the entire image domain. It equals or approaches the value one at the object boundaries and approaches the value zero where the image gradient is small. One of the methods is a simple extension of the formulation of Kimia et al [3,9,19] in which the velocity of the curve is multiplied by a "stopping term", $(1 - v)$, so that the evolution is slowed down near the object boundaries (in fact stopped where $v = 1$). In an alternate approach [4,8,14,15], the idea is to let the curve evolve towards a geodesic in the metric defined by $(1 - v)^q$ where q is a constant, usually equal to 1 or 2. The corresponding point velocity of the curve along its normal consists of the curvature term as before and an advection term given by the derivative of v in the direction of the normal. There is no constant component corresponding to morphology. There is no stopping term either. The easiest way to implement curve evolution is by embedding the intial curve as a level curve in a surface and let all the level curves of the surface evolve simultaneously. The advantage is that changes in the topology of the curve are handled automatically, simplifying the data structure. Numerical scheme of Osher and Sethian [11] may then be used to implement the evolution.

It turns out that the different forms of curve evolution described above including the choice of initial curve, the choice of the edge-strength function and the embedding of the curve in a surface, can be incorporated into a single segmentation functional as described in [16]. An added bonus is that there is no restriction on the evolving curve so that multiple intersecting curves are permitted. Another bonus is that the set of level curves of the associated edge-strength function constitutes a scale space for shapes analogous to the entropy scale space proposed in [7]. In particular, various kinds of shocks and resulting shape skeleton may computed quite easily from v [18]. The objective of this paper is to generalize this functional to the case of vector-valued images, in particular color images.

2. The New Segmentation Functional (scalar case)

The new segmentation functional from [16] is as follows:

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$$E(u,B) = \int\int_{R\setminus B} ||\nabla u|| dxdy + \frac{\beta}{\alpha}\int\int_R |u-g|dxdy + \int_B \frac{J_u}{1+\alpha J_u} ds \tag{1}$$

where R is a connected, bounded, open subset of $\mathbf{R}^2$, g is the feature intensity, B is a curve segmenting R, u is the smoothed image $\subset \mathbf{R}^2 \setminus B$ and J_u is the jump in u across B, that is, $J_u = |u^+ - u^-|$ where the superscripts + and – refer to the values on two sides of B. α, β are the weights. Note that each boundary point is assigned weight according to its level of contrast instead of being assigned a fixed weight. The associated approximation which is the actual functional used for implementation by gradient descent is:

$$E_\rho(u,v) = \int\int_R \{\alpha(1-v)^2 ||\nabla u|| + \beta|u-g| + \frac{\rho}{2}||\nabla v||^2 + \frac{v^2}{2\rho}\} dxdy \tag{2}$$

The gradient descent equations for $E_\rho(u,v)$ are:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -2\nabla v \cdot \nabla u + (1-v)||\nabla u||\ curv(u) \\ &\quad - \frac{\beta}{\alpha(1-v)}||\nabla u|| \frac{(u-g)}{|u-g|} \\ \frac{\partial v}{\partial t} &= \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1-v)||\nabla u|| \\ &\frac{\partial u}{\partial n}|_{\partial R} = 0;\ \frac{\partial v}{\partial n}|_{\partial R} = 0 \end{aligned} \tag{3}$$

$$curv(u) = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{\left(u_x^2 + u_y^2\right)^{3/2}} \tag{4}$$

$curv(u)$ is the curvature of the level curves of u. The edge-strength function v may be thought of as (non-linearly) smoothed and normalized gradient of u.

Although functionals (1) and (2) superficially look very similar to the Mumford-Shah functional [10] and its approximation by Ambrosio and Tortorelli [2], they behave in a fundamentally different way:

Curve Evolution: The level curves of u move with point velocity of

$$2\nabla v \cdot \frac{\nabla u}{||\nabla u||} - (1-v)curv(u) + \frac{\beta}{\alpha(1-v)} \frac{u-g}{|u-g|} \tag{5}$$

in the direction of the gradient of u. The first term is advection, pulling the level curve towards higher values of v. The second term smooths the level curve as described in [1]. The last term moves the level curve with a point velocity of $\pm\beta/\alpha(1-v)$. The sign is automatically chosen such that this component of velocity pushes the level curve towards the corresponding level curve of g. If v is approximately constant along each level curve, then the last term may be seen to correspond to the constant velocity component used by Kimia et al. Thus the equation may be seen as combining all three types of velocity components described in the introduction. The ratio β/α may be interpreted as the smoothing radius for the level curves: If v is set identically equal to zero, the equation smooths level curves such that their radius of curvature is not less than β/α anywhere.

Shocks: The most important property of $E_\rho(u,v)$ is that u develops shocks and thus object boundaries are recovered *as actual discontinuities*. The reason is that the evolution equation for u is parabolic only along the level curves of u; it is *hyperbolic* in the direction normal to the level curves. Note however that the edge-strength function, v, is still a continuous function and hence, the actual boundaries are to be recovered from the discontinuities of u.

Deblurring: To illustrate the deblurring capability of the new functional, consider the case of a 1–dimensional image domain R. Note that, in contrast to other formulations involving curve evolution, the new functional can be applied even to the one dimensional case. When R is one dimensional, the $curv(u)$ term drops out and the evolution equation for u becomes purely hyperbolic, governed by advection due to v and the "constant" velocity component. Assume that g is zero in the first third of R, increases linearly to 1 in the second and stays equal to 1 in the third. When the diffusion system (3) is applied to this example, v develops a unique maximum at the center of R, inducing motion of points on the graph of u towards the center of R, eventually producing a shock there. The steady state solution is a piecewise constant function with a unique discontinuity at the center of R. Since the penalty is higher for two breaks than a single large break combining the two, *we do not get multiple breaks* along the ramp.

3. The New Segmentation Functional (vector case)

Assume now that the image g is a set of m functions: $g = \{g_1, ..., g_m\}$. Since the focus of this paper is on implementation, only the generalization of E_ρ is considered below. If the components of g are unrelated functions, then the simplest way to link their evolution is through a common edge-strength function. That is, consider

$$E_\rho(u,v) = \int\int_R \{(1-v)^2 \left(\sum \alpha_k \|\nabla u_k\|\right) \tag{6}$$
$$+ \sum \beta_k |u_k - g_k| + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho}\} dxdy$$

with associated gradient descent equations:

(7)
$$\frac{\partial u_i}{\partial t} = -2\nabla v \cdot \nabla u_i + (1-v)\|\nabla u_i\| \; curv(u_i)$$
$$- \frac{\beta_i}{\alpha_i(1-v)}\|\nabla u_i\| \frac{(u_i - g_i)}{|u_i - g_i|} \quad i = 1, \cdots, m$$
$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{v}{\rho^2} + \frac{2}{\rho}(1-v)\sum \alpha_k \|\nabla u_k\|$$
$$\frac{\partial u_i}{\partial n}|_{\partial R} = 0; \; \frac{\partial v}{\partial n}|_{\partial R} = 0$$

The level curves of each component u_i evolve just as before, independently of the other components except for the common constraint imposed by their common edge-strength function.

A more interesting evolution is obtained if g is a vector-valued function so that it is more appropriate to use the total variation of vector-valued functions. That is, consider the functional:

$$E_\rho(u,v) = \int\int_R \{\alpha(1-v)^2 \|\nabla u\| \tag{8}$$
$$+ \frac{\beta}{\sqrt{m}} \sum |u_k - g_k| + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho}\} dxdy$$

where $\|\nabla u\| = \sqrt{\sum \|\nabla u_k\|^2}$. The factor $\sqrt{m}$ is introduced so that the functional reduces exactly to the scalar case when $g_1 = g_2 = \ldots = g_m$. Alternatively, the term $\Sigma|u_k - g_k|/\sqrt{m}$ may be replaced by $\sqrt{\Sigma|u_k - g_k|^2}$. (Of course, instead of the Euclidean norm, any other norm induced by a positive definite quadratic form may be used.)

The gradient descent equations are:

(9)
$$\frac{\partial u_i}{\partial t} = -2\nabla v \cdot \nabla u_i$$
$$+ (1-v)\left\{\nabla \cdot \nabla u_i - \frac{\nabla u_i \cdot \sum H_k \nabla u_k}{\|\nabla u\|^2}\right\}$$
$$- \frac{\beta_i}{\sqrt{m}\alpha_i(1-v)}\|\nabla u\| \frac{(u_i - g_i)}{|u_i - g_i|} \quad i = 1, \cdots, m$$
$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1-v)\|\nabla u\|$$
$$\frac{\partial u_i}{\partial n}|_{\partial R} = 0; \; \frac{\partial v}{\partial n}|_{\partial R} = 0$$

Here, H_k is the hessian of u_k. More explicitly,

$$\nabla u_i \cdot H_k \nabla u_k = u_{i,x} u_{k,x} u_{k,xx} + u_{i,x} u_{k,y} u_{k,xy} \tag{10}$$
$$+ u_{i,y} u_{k,x} u_{k,xy} + u_{i,y} u_{k,y} u_{k,yy}$$

4. Illustrative Example

Equations (9) were tested on the 128×128 (pseudo)color medical image[2] (cryosections) shown in the figure below. The top row shows the red, green and blue components of the image, with pixel values ranging from 0 to 120. (Each component is displayed after scaling it to a common pixel range of 0 to 255.) Following [13], the RGB components where transformed into CIE 1976 $L^* a^* b^*$ space which behaves more like a vector space. The values of ρ and α/β were set equal to 4 and 8 pixels respectively. The necessary value of α depends on the level of contrast across the boundary of interest. If the value is too low, the value of v would be much less than one and features of interest might be smoothed away. If the value is too high, diffusion is stopped too early and the result is a noisy boundary. For the purpose of this example, two separate values of α were chosen, one for detecting the high contrast boundary surrounding the two cross-sections (the middle row in the figure) and a higher value to detect the weaker boundaries of some of the internal structures (the bottom row). Due to the presence of shocks, the first and third term of the evolution equation have to be calculated carefully. The formulae of Osher and Sethian were used for each of these terms. As mentioned before, the boundaries must be recovered as discontinuities in u. To show how well the common discontinuities develop for the three components, the boundaries were recovered as level curves of the L^*– component and are shown superposed on the three original RGB components.

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