Handout - Integration a.k.a. Antiderivatives

Formulas

$f(x)$	$\frac{df}{dx}$	$\int f\,dx$
$a \cdot x^n,\ n \neq -1$	$a \cdot n \cdot x^{n-1}$	$\frac{a}{n+1} x^{n+1} + C$
$\frac{a}{x}$	$-x^{-2} = \frac{-a}{x^2}$	$a \cdot \ln(\lvert x \rvert) + C = ln(\lvert x \rvert^a) + C$
$a \cdot x$	a	$a \cdot \frac{x^2}{2} + C$
a	0	$a \cdot x + C$
$a \cdot \sin(x)$	$a \cdot \cos(x)$	$-a \cdot \cos(x) + C$
$a \cdot \cos(x)$	$-a \cdot \sin(x)$	$a \cdot \sin(x) + C$
$a \cdot \tan(x)$	$a \cdot \sec^2(x)$	$a \cdot \ln(\lvert \sec(x) \rvert) + C$
$a \cdot e^x$	$a \cdot e^x$	$a \cdot e^x + C$
$a \cdot \ln(x)$	$\frac{a}{x}$	$a \cdot x \ln(x) - a \cdot x + C$
$a \cdot f \pm b \cdot g$	$a \cdot f' \pm b \cdot' g$	$a \cdot F \pm b \cdot G + C$

Indefinite Integrals *The answers are functions.*

Example 1a. $\int 6 \cdot x^2 \, dx$

$a = 6$
$n = 2, \quad \Rightarrow n + 1 = 3.$

$$\int 6 \cdot x^2 dx = 6 \cdot \frac{x^3}{3} + C$$

$$\boxed{y = 2 \cdot x^3 + C}$$

Example 1b. $\int 5 \cdot \sqrt{x} \, dx$

$a = 5$
$n = \frac{1}{2}, \quad \Rightarrow n + 1 = \frac{3}{2}.$

$$\int 5 \cdot x^{1/2} \, dx = 5 \cdot \frac{x^{3/2}}{3/2} + C$$

$$\boxed{y = \tfrac{10}{3} \cdot x^{3/2} + C = \tfrac{10}{3} \cdot \sqrt{x^3} + C}$$

Example 1c. $\int 6\cos(x) - \frac{7}{x} + 3e^x \, dx$

Integrate each term in the sum by its own rule.

$$\int 6\cos(x) - \frac{7}{x} + 3e^x \, dx =$$

$$\boxed{y = 6\sin(x) - 7 \cdot ln(|x|) + 3e^x + C}$$

Solving for C

Example 2a. Solve the Initial Value Problem $y' = 8x - 5,\ y(0) = 3$

Note: $y(0) = 3$ is the initial value. It means $y = 3$ when $x = 0$.

Steps

(1) Integrate the expression $8x - 5$
(2) Plug in the point $(x, y) = (0, 3)$
(3) Solve for C

1) $y = \int 8x - 5\,dx = 4x^2 - 5x + C$

2) $y = 4x^2 - 5x + C$
$3 = 4(0)^2 - 5(0) + C$

3) $\Rightarrow\ C = 3$

$$\boxed{y = 4x^2 - 5x + 3}$$

Example 2b. Solve the Initial Value Problem $y' = 10 - 6x,\ y(-1) = -6$

1) $y = \int 10 - 6x\,dx = 10x - 3x^2 + C$

2) $y = -3x^2 + 10x + C$
$-6 = -3(-1)^2 + 10(-1) + C$

3) $\Rightarrow\ C = 7$

$$\boxed{y = -3x^2 + 10x + 7}$$

Definite Integrals *The Answers are Numbers*

Example 3a. Calculate the Definite Integral $\int_{-1}^{2} x^2 - 3x\,dx$

Steps

(1) Integrate the expression (*ignore C*)
(2) Plug in the two x-values
(3) Subtract

$$1)\ F(x) = \int x^2 - 3x\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2$$

$$2)\quad F(2) = \frac{1}{3}(2)^3 - \frac{3}{2}(2)^2 = \frac{8}{3} - 6 = -\frac{10}{3}$$

$$F(-1) = \frac{1}{3}(-1)^3 - \frac{3}{2}(-1)^2 = -\frac{1}{3} - \frac{3}{2} = -\frac{11}{6}$$

$$3)\quad \Rightarrow F(2) - F(-1) = -\tfrac{10}{3} - \left(-\tfrac{11}{6}\right)$$

$$\boxed{-\frac{3}{2}}$$

Example 3b. Calculate the Definite Integral $\int_{0}^{\pi} \sin(x)\,dx$

$$1)\ F(x) = \int \sin(x)\,dx = -\cos(x)$$

$$2)\quad F(\pi) = -\cos(\pi) = -(-1) = +1$$

$$F(0) = -\cos(0) = -1$$

$$3)\quad \Rightarrow F(\pi) - F(0) = 1 - (-1)$$

$$\boxed{2}$$

If you use a calculator for trig. functions - set it to 'radian' mode.

Example 3c. *Calculate the Definite Integral* $\int_5^{35} \frac{1}{x}\, dx$

1) $F(x) = \int \frac{1}{x}\, dx = \ln(x)$

2) $F(35) = \ln(35)$

$F(5) = \ln(5)$

3) $\Rightarrow F(35) - F(5) = \ln(35) - \ln(5)$

$= \ln(35/5)$

$\boxed{\ln(7)}$

Algebraic Rule 1 for Log functions: $\ln(a) + \ln(b) = \ln(a \cdot b)$

Algebraic Rule 2 for Log functions: $\ln(a) - \ln(b) = \ln(a/b)$

Algebraic Rule 3 for Log functions: $a \cdot \ln(b) = \ln(b^a)$

u-Substitution

Example 4a. Determine the Indefinite Integeral $\displaystyle\int (2x+1)e^{x^2+x}\,dx$

ISSUE: We do not have a formula for this. The closest formula we have is $\int e^x\,dx = e^x + C$.

Steps:

(1) Substitute for the expression: $u = x^2 + x$
(2) Differentiate and solve for dx
(3) Substitute for u and dx in the integral
(4) Cancel out the "x" terms
(5) Integrate the u expression
(6) Substitute back.

1) $u = x^2 + x$

2) $\frac{du}{dx} = 2x + 1 \quad \Rightarrow dx = \frac{du}{2x+1}$

3) $\int (2x+1)e^{x^2+x}\,dx = \int (2x+1)e^u\,\frac{du}{2x+1}$

4) $= \int e^u\,du$

5) $= e^u + C$

6) $= e^{x^2+x} + C$

$$\boxed{e^{x^2+x} + C}$$

Example 4b. Calculate $\displaystyle\int 4x\sqrt{x^2+1}\,dx$

1) $u = x^2 + 1$
2) $\frac{du}{dx} = 2x \quad \Rightarrow dx = \frac{du}{2x}$
3) $\int 4x\sqrt{x^2+1}\,dx = \int 4x\sqrt{u}\,\frac{du}{2x}$
4) $= \int 2\sqrt{u}\,du$
5) $= 2 \cdot \frac{2}{3}u^{3/2} + C$
6) $= \frac{4}{3}\sqrt{x^2+1} + C$

$$\boxed{\frac{4}{3}\sqrt{x^2+1} + C}$$

Example 4c. Calculate $\displaystyle\int (6x^3+6x^2)(3x^4+4x^3-17)^5\,dx$

1) $u = 3x^4 + 4x^3 - 17$
2) $\frac{du}{dx} = 12x^3 + 12x^2 \quad \Rightarrow dx = \frac{du}{12x^3+12x^2}$
3) $\int (6x^3+6x^2)(3x^4+4x^3-17)^5\,dx = \int (6x^3+6x^2)(u)^5\,\frac{du}{12x^3+12x^2}$
4) $= \int \frac{1}{2}u^5\,du$
5) $= \frac{1}{2} \cdot \frac{1}{6}u^6 + C$
6) $= \frac{1}{12}(3x^4+4x^3-17)^6 + C$

$$\boxed{\frac{1}{12}(3x^4+4x^3-17)^6 + C}$$

Exercises - Integration a.k.a. Antiderivatives

Calculate the Indefinite Integrals

a) $\displaystyle\int 90x^9 + 54x^5 + 22\,dx$

b) $\displaystyle\int 35x^4 + 12x^2 + 88\,dx$

c) $\displaystyle\int 2x^{53/6} - 6x^{29/6} + 78\,dx$

d) $\displaystyle\int 7x^{34/5} + 3x^{19/5} - 30\,dx$

e) $\displaystyle\int \frac{81}{x^8} + \frac{36}{x^5} - 87\,dx$

f) $\displaystyle\int \frac{49}{x^6} - \frac{32}{x^3} + 68\,dx$

g) $\displaystyle\int -\frac{1}{x^{3/4}} + \frac{5}{x^{7/4}} - 31\,dx$

h) $\displaystyle\int -\frac{1}{x^{4/7}} + \frac{4}{x^{18/7}} - 46\,dx$

i) $\displaystyle\int -\frac{5}{x^3} - \frac{87}{x} + 4x^3\,dx$

j) $\displaystyle\int \frac{7}{x^3} - \frac{45}{x} + 3x^3\,dx$

Answers a) $9x^{10} + 9x^6 + 22x + C$; b) $7x^5 + 4x^3 + 88x + C$;

c) $2\left(\frac{6x^{59/6}}{59} - \frac{18x^{35/6}}{35} + 39x\right) + C$; d) $\frac{35x^{39/5}}{39} + \frac{5x^{24/5}}{8} - 30x + C$;

e) $-\frac{81}{7x^7} - \frac{9}{x^4} - 87x + C$; f) $-\frac{49}{5x^5} + \frac{16}{x^2} + 68x + C$;

g) $-4\sqrt[4]{x} - \frac{20}{3x^{3/4}} - 31x + C$; h) $-\frac{7x^{3/7}}{3} - \frac{28}{11x^{11/7}} - 46x + C$;

i) $\frac{5}{2x^2} + -87\ln(x) + x^4 + C$; j) $-\frac{7}{2x^2} + -45\ln(x) + \frac{3x^4}{4} + C$;

Evaluate the Definite Integrals

a) $\displaystyle\int_{-1}^{1} 19x^5\,dx$

b) $\displaystyle\int_{-3}^{1} 19x^2\,dx$

c) $\displaystyle\int_{-\frac{3}{5}}^{\frac{2}{5}} 19x^5\,dx$

d) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{4}} 15x^2\,dx$

e) $\displaystyle\int_{1}^{3} x^4 - 17x\,dx$

f) $\displaystyle\int_{0}^{2} x^5 - 11x\,dx$

g) $\displaystyle\int_{2}^{16} \frac{12}{x}\,dx$

h) $\displaystyle\int_{2}^{16} \frac{12}{x}\,dx$

Solve the Initial Value Problems.

a) $y' = 4x - 8, \quad y(2) = 0$

b) $y' = 2x, \quad y(0) = 7$

c) $y' = 4x + 5, \quad y(\frac{2}{3}) = \frac{47}{9}$

d) $y' = 9x^2 - 6, \quad y(\frac{5}{2}) = \frac{223}{8}$

Answers a) 0; b) $\frac{532}{3}$; c) $-\frac{2527}{18750}$; d) $\frac{45}{64}$; e) $-\frac{98}{5}$; f) $-\frac{34}{3}$; g) $12\ln(8) = \ln\left(8^{12}\right) = 24.9533$; h) $12\ln(8) = \ln\left(8^{12}\right) = 24.9533$;

Answers: a) $y = 2x^2 - 8x + 8$; b) $y = x^2 + 7$; c) $y = 2x^2 + 5x + 1$; d) $y = 3x^3 - 6x - 4$;

Evaluate the Indefinite Integrals

a) $\displaystyle\int (56x+42)\left(4x^2+6x+11\right)^6\,dx$

c) $\displaystyle\int \frac{80x^3+80x}{\sqrt{4x^4+8x^2+11}}\,dx$

b) $\displaystyle\int (160x+128)\left(5x^2+8x+13\right)^3\,dx$

d) $\displaystyle\int \frac{40x+40}{(4x^2+8x+9)^{2/3}}\,dx$

e) $\displaystyle\int (24x+8)e^{3x^2+2x-5}dx$

f) $\displaystyle\int (16x-8)e^{4x^2-4x-11}dx$

g) $\displaystyle\int \frac{10x+20}{x^2+4x+6}\,dx$

h) $\displaystyle\int \frac{32x^3+24x}{2x^4+3x^2+8}\,dx$

i) $\displaystyle\int -480x^2\cos(18-4x^3)\sin^7(18-4x^3)dx$

j) $43\left(35x^4-1\right)\sin\left(7x^5-x+12\right)$

Answers a) $\left(4x^2+6x+11\right)^7+C$; b) $4\left(5x^2+8x+13\right)^4+C$; c) $10\sqrt{4x^4+8x^2+11}+C$;

d) $15\sqrt[3]{4x^2+8x+9}+C$; e) $4e^{3x^2+2x-5}+C$; f) $2e^{4x^2-4x-11}+C$;

g) $5\ln\left(x^2+4x+6\right)+C$; h) $4\ln\left(2x^4+3x^2+8\right)+C$;

i) $5\sin^8(18-4x^3)+C$; j) $-43\cos\left(7x^5-x+12\right)+C$;