2/23/2024

On the geometry and Topology of polynomials

1. Roots of polynomials.

- A polynomial (in a single variable x) of degree n with coefficients in $\mathbb{C}$

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

$(a_i \in \mathbb{C}, a_0 \neq 0)$

- The roots of P are the solutions to the (polynomial) equation $P(x) = 0$.

- To solve such an equation, we may as well divide P by a_0 and only consider monic polynomials,

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

- By the Fundamental Theorem of Algebra, every non-constant polynomial completely factors into a product of linear factors:

$$\boxed{P(x) = (x - z_1) \cdots (x - z_n)}$$

where $z_1, \ldots, z_n$ are the roots of P.

- The coefficients of P may be recovered from its roots, as the elementary symmetric polynomials in those roots (up to sign):

$$\begin{cases} a_1 = -(z_1 + z_2 + \cdots + z_n) \\ a_2 = \sum_{i<j} z_i z_j \\ \vdots \\ a_n = (-1)^n z_1 \cdots z_n \end{cases}$$

- These formulas were discovered by François Viète in the late 1500s.

2. Tschirnhaus transformations

- Linear equations:

$$x + a_0 = 0 \longrightarrow x = -a_0$$

- Qudratic equations (complete the square)

[Babylonians]

$$x^2 + a_1 x + a_2 = 0$$

Set $y = x + \frac{a_1}{2} \longrightarrow \left(y - \frac{a_1}{2}\right)^2 + a_1\left(y - \frac{a_1}{2}\right) + a_2 = 0$

$$\longrightarrow y^2 - \frac{a_1^2}{4} + a_2 = 0 \longrightarrow y = \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

$$\therefore x = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

- In general, we may get rid of the coefficient of x^{n-1} by making the substitution

$$y = x + \frac{a_1}{n}$$

$$\to x^n + a_1 x^{n-1} + \cdots = \left(y - \frac{a_1}{n}\right)^n + a_1\left(y - \frac{a_1}{n}\right)^{n-1} + \cdots$$

$$= \left(y^n - a_1 y^{n-1} + \cdots\right) + \left(a_1 y^{n-1} - \cdots\right) + \cdots$$

$$= y^n + \left(a_2 - \frac{n-1}{2n} a_1^2\right) y^{n-2} + \cdots$$

- Thus, it is enough to consider "depressed" polynomials,

$$P(x) = x^n + a_{n-2} x^{n-2} + \cdots + a_{n-1} x + a_n$$

- How to solve the depressed cubic equation,

$$\boxed{P(x) := x^3 + a_2 x + a_3 = 0}$$

- We perform a quadratic substitution, due to Tschirnhaus [1638]

$$\boxed{y = s(x) = x^2 + b_2 x + b_3}$$

- Then y also satisfies a monic cubic polynomial,

$$\boxed{Q(y) := y^3 + c_1 y^2 + c_2 y + c_3 = 0}$$

- We may compute the coefficients of Q using Vieta's formulas

- Let x_1, x_2, x_3 be the roots of P
  $y_i = s(x_i)$ the roots of Q
- Then:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 x_2 + x_1 x_3 + x_2 x_3 = a_2 \\ x_1 x_2 x_3 = -a_3 \end{cases} \qquad \begin{aligned} &y_1 + y_2 + y_3 = -c_1 \\ &y_1 y_2 + y_1 y_3 + y_2 y_3 = c_2 \\ &y_1 y_2 y_3 = -c_1 \end{aligned}$$

- Hence:

$$\begin{aligned} c_1 &= -\left(s(x_1) + s(x_2) + s(x_3)\right) \\ &= -\left(x_1^2 + x_2^2 + x_3^2 + b_2 (x_1 + x_2 + x_3) + 3b_3\right) \\ &= -\left[(x_1 + x_2 + x_3)^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3) + b_2 (x_1 + x_2 + x_3) + 3b_3\right] \\ &= -\left[0 - 2a_2 + b_2 \cdot 0 + 3b_3\right] \\ &= 2a_2 - 3b_3 \end{aligned}$$

- Similarly:

$$c_2 = a_2 b_2^2 + a_2^2 + a_3 b_2 - 4a_2 b_3 + 3b_3^2$$

$$\begin{aligned} c_3 = {} & a_3 b_2^3 - a_2 b_2^2 b_3 + a_2 a_3 b_2 - a_2^2 b_3 - 3a_3 b_2 b_3 \\ & + 2a_2 b_3^2 - b_3^3 - a_3^2 \end{aligned}$$

- The goal is to choose b_2 & b_3 such that the equation $Q(y) = 0$ will depend on a single parameter, i.e.: $c_1 = c_2 = 0$.
- We get:

$$b_3 = \frac{2}{3} a_2$$

$$b_2 = \frac{-3a_3 - 6\sqrt{\frac{a_3^2}{4} + \frac{a_2^3}{27}}}{2a_2}$$

- Plugging these values into c_3 and solving the equation $y^3 = -c_3$, we get 3 roots

$$Y_1 = \frac{6}{a_2}\sqrt{\left(\frac{a_3}{2}\right)^2 + \left(\frac{a_2}{3}\right)^3} \cdot \sqrt[3]{-\frac{a_3}{2} + \sqrt{\left(\frac{a_3}{2}\right)^2 + \left(\frac{a_2}{3}\right)^3}}, \quad Y_2 = Y_1 \cdot \zeta, \quad Y_3 = Y_1 \zeta^2$$

where $\zeta = e^{\pi i/3} = \frac{-1 + i\sqrt{3}}{2}$.

- Finally, we go back to the equation $x^2 + b_2 x + (b_3 - y) = 0$ and solve for x.

- In the end, the 3 roots of $P(x) = 0$ are:

$$x_1 = u + v, \quad x_2 = u\zeta + v\zeta^2, \quad x_3 = u\zeta^2 + v\zeta$$

where

$$u = \sqrt[3]{-\frac{a_3}{2} + \sqrt{\left(\frac{a_3}{2}\right)^2 + \left(\frac{a_2}{3}\right)^3}}$$

$$v = \sqrt[3]{-\frac{a_3}{2} - \sqrt{\left(\frac{a_3}{2}\right)^2 + \left(\frac{a_2}{3}\right)^3}}$$

are chosen so that $uv = -\frac{a_3}{2}$.

- These are Carolano's formulas, which he attributed to del Ferro and Tartaglia
- A similar method solves the quartic (Ferrari)
- The method breaks down for the quintic, which cannot be solved by radicals alone: Ruffini (1799–1813), Abel (1824), Galois (1832)

3. The Bring radical

- Using Tschirnhaus transformations, a general quintic equation can be brought to the form
$$P(x) := x^5 + a_3 x^2 + a_4 x + a_5 = 0.$$
- As shown by Bring (1796) and Jerrard (1852) one can eliminate one more parameter and bring it to the form
$$y^5 + b_4 y + b_5 = 0$$
via a cubic Tschirnhaus transformation.
- The Bring–Jerrard form can be further reduced by setting $z = y(-b_4)^{-1/4}$. We get:
$$\boxed{z^5 - z + a = 0}$$
where $= b_5 (-b_4)^{-5/4}$.
- The <u>Bring radical</u> of a, denoted $\mathrm{Br}(a)$, is any one of the 5 solutions of this equation
- Upshot: One can solve the quintic by using the field operations on $\mathbb{C}$ and the functions $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, and Br. For short: the quintic can be solved by radicals and the Bring radical.

4. Hilbert's Thirteenth problem

- A general polynomial equation of degree n can be brought to the form:

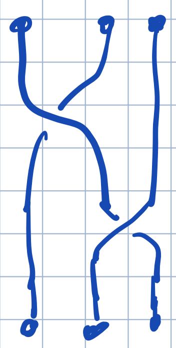
$$\boxed{P(x) := x^n + b_1 x^{n-4} + \dots + b_{n-4} x + 1 = 0}$$

- Thus, the solutions to such equations can be expressed in terms of functions of at most $n-4$ variables.
- For $n \le 5$, those functions are $\sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }$, Br.
- Hilbert asked (at ICM 1900) whether this is an optimal bound, or whether the number of variables can be decreased.
- Define the <u>resolvent degree</u>

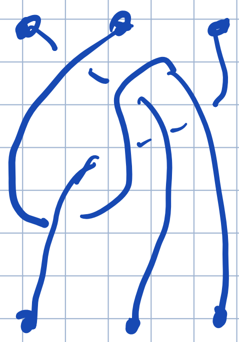
$$\boxed{RD(n) = \min\left\{ d \;\middle|\; \begin{array}{l} \exists \text{ formula using only} \\ \text{algebraic functions} \\ \text{of degree} \le d \end{array} \right\}}$$

- $RD(n) = 1$ for $n \le 5$
- Hilbert: What is $RD(n)$ for $n = 6, 7, \dots$?

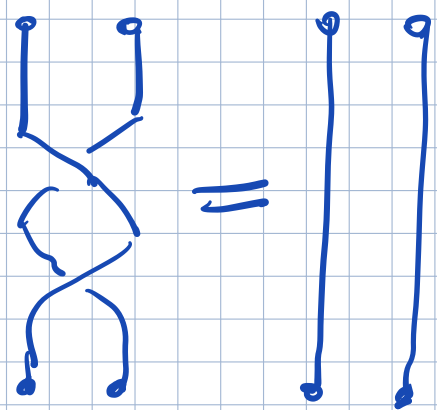
5. Braid groups



Braid on 3 strands

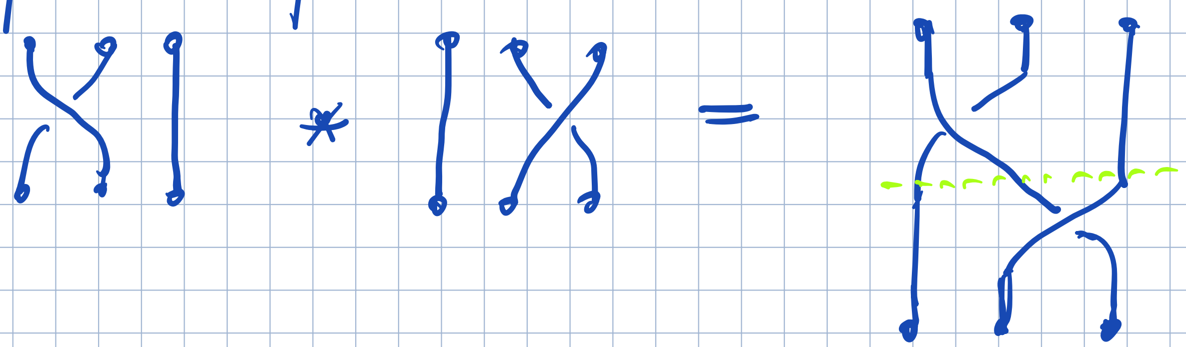


Not a braid



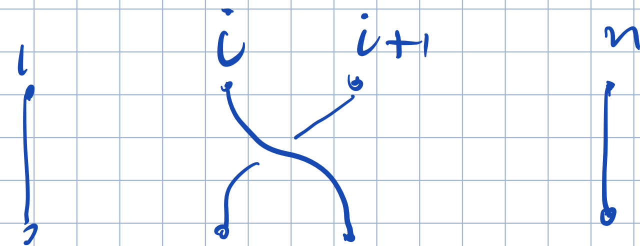
Same braids (isotopic)

- Composition of braids:



- With this operation, the set of all braids on n strands, B_n, becomes a group.
- Identity: 
- Inverse: reversed braid 

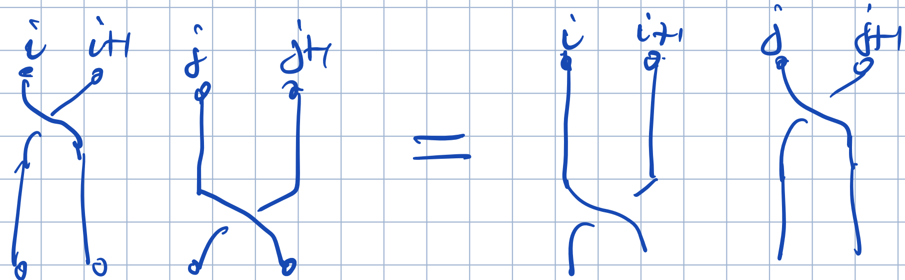
- The n-th braid group B_n is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$:



σ_i

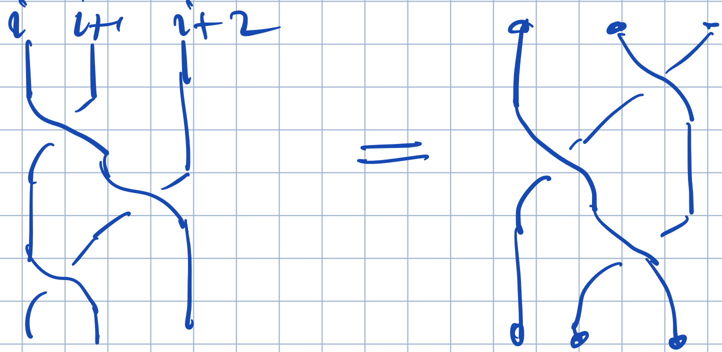
Subject to the relations

(1)



$\sigma_i \cdot \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

(2)



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

- E.g.: $B_2 = \{\sigma_1^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$

$B_4 = \left\langle \sigma_1, \sigma_2, \sigma_3 \;\middle|\; \begin{array}{l} \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \\ \sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3 \end{array} \quad \sigma_1 \sigma_3 = \sigma_3 \sigma_1 \right\rangle$

6. Ordered configurations

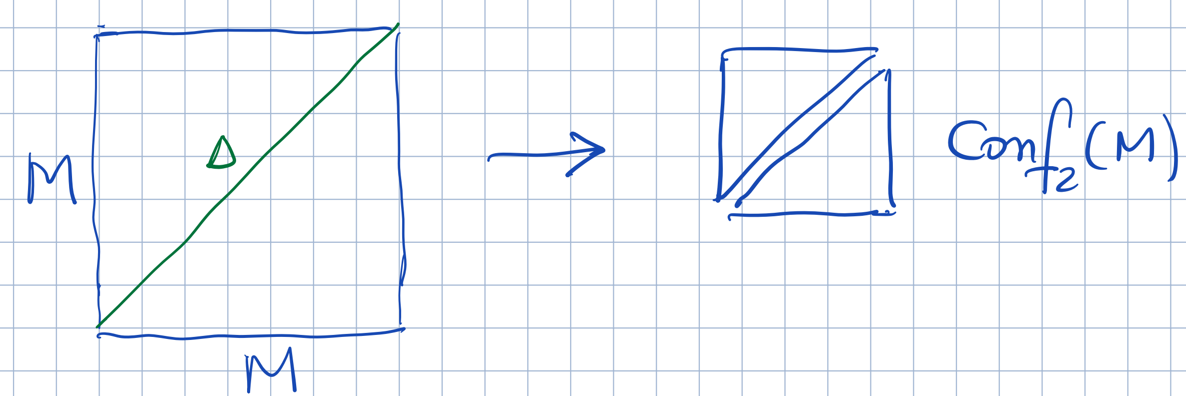
- Let M be a topological space.
- For each $n \geq 1$ let $M^n = \underbrace{M \times \cdots \times M}_{n \text{ times}}$ (with product topology).

Definition The <u>ordered configuration space</u> of n points in M is the space

$$\mathrm{Conf}_n(M) = \{(m_1, \dots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\}$$

- If M is a manifold of dimension d, then $\mathrm{Conf}_n(M)$ is a manifold of dimension dn

Eg:
- $\mathrm{Conf}_1(M) = M$
- $\mathrm{Conf}_2(M) = M \times M \setminus \Delta$ (Δ = diagonal)



- Suppose M is connected. Then
  - If $d = 1$ and $n \geq 2$, then $\mathrm{Conf}_n(M)$ is disconnected.
  - If $d \geq 2$, then $\mathrm{Conf}_n(M)$ is connected.

- Suppose now $M = G$ is a topological group (e.g., $G = \mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, S^1, etc)

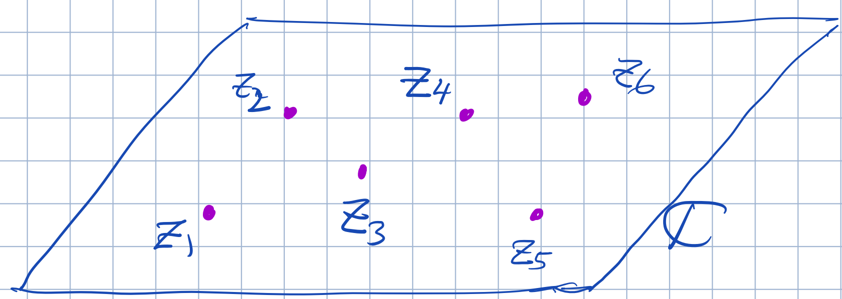
Then

$$\mathrm{Conf}_n(G) \xrightarrow{\cong} G \times \mathrm{Conf}_{n-1}(G \setminus \{e\})$$

$$(g_1, \dots, g_n) \longrightarrow (g_1, (g_1^{-1} g_2, \dots, g_1^{-1} g_n))$$

- Most important case is when $M = \mathbb{R}^2$, i.e., $M = \mathbb{C}$.

- A point in $\mathrm{Conf}_n(\mathbb{C})$ can be viewed as an n-tuple of distinct points in $\mathbb{C}$



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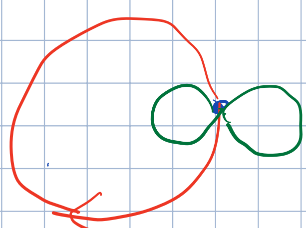
  - $\mathrm{Conf}_1(\mathbb{C}) = \mathbb{C} \simeq \{0\}$
  - $\mathrm{Conf}_2(\mathbb{C}) \cong \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \simeq \{0\} \times S^1 \simeq S^1$
  - $\mathrm{Conf}_3(\mathbb{C}) \cong \mathbb{C} \times \mathrm{Conf}_2(\mathbb{C} \setminus \{0\})$

$$\cong \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0, 1\})$$



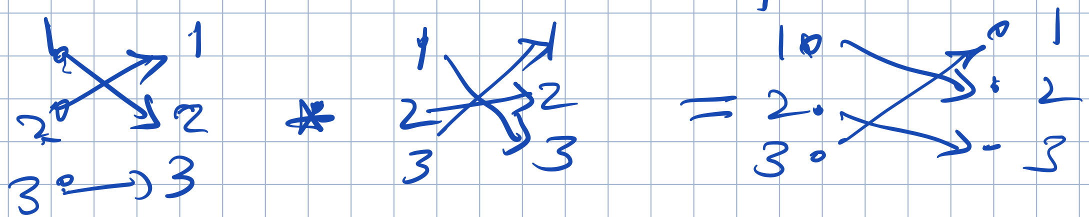
$$\simeq S^1 \times (S^1 \vee S^1)$$

7. Unordered configurations

- A *permutation* of a set S is a bijection from S to S.

- For instance, if $S = \{1, 2\}$, its permutations are
$$\begin{matrix} 1 \to 1 \\ 2 \to 2 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 \to 2 \\ 2 \to 1 \end{matrix}$$

- Permutations can be composed. E.g.:



- The *symmetric group* S_n is the group of all permutations of the set $\{1, \ldots, n\}$. Its order is $n!$

- Given a space M, the group acts on M^n by permuting the coordinates:
$$S_n \times M^n \longrightarrow M^n$$
$$(\sigma, (m_1, \ldots, m_n)) \longmapsto (m_{\sigma(1)}, \ldots, m_{\sigma(n)})$$

- This action restricts to a *free* action of S_n on $\mathrm{Conf}_n(M)$.

Definition The unordered configuration space of n points in a space M is

$$\mathrm{UConf}_n(M) = \mathrm{Conf}_n(M)/S_n$$

(the quotient space of the ordered configuration space by the above free action).

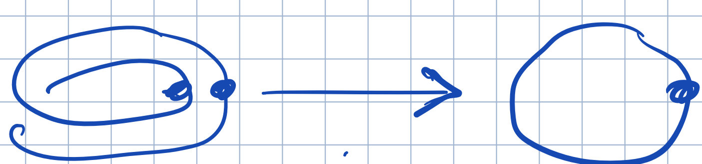
- $\mathrm{UConf}_n(M)$ may be viewed as the space of all subsets $\{m_1, \dots, m_n\} \subset M$ of size n.
- The quotient map,

$$\mathrm{Conf}_n(M) \longrightarrow \mathrm{UConf}_n(M),$$

is a (regular) cover, with group of deck transformations S_n.

- E.g., $\mathrm{Conf}_2(\mathbb{C}) \longrightarrow \mathrm{UConf}_2(\mathbb{C})$ is, up to homotopy, the 2-fold cover

$$S^1 \longrightarrow S^1 \qquad z \longmapsto z^2$$

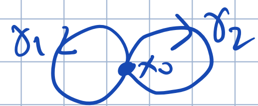


8. Fundamental groups of configuration spaces

- Fundamental group of M, based at x_0:

$$\pi_1(M, x_0) = \left\{ \begin{array}{l} \text{loops } \gamma \text{ in } M \text{ based at } x_0 \\ \text{modulo homotopy rel } x_0 \end{array} \right\}$$

with
- $\gamma_1 \cdot \gamma_2$ = concatenation of the two loops



- γ^{-1} = loop traversed in opposite direction
- identity = constant loop at x_0

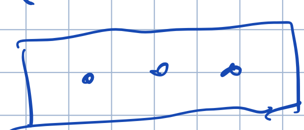
- If M is path-connected, then

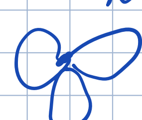
$$\pi_1(M, x_0) \cong \pi_1(M, x_1) \quad \text{for all } x_0, x_1 \in M$$

so write it simply as $\pi_1(M)$.

- $\pi_1(M_1 \times M_2) \cong \pi_1(M_1) \times \pi_1(M_2)$

- Examples:
  - $\pi_1(\text{contractible space}) = \{e\}$ — e.g., $\mathbb{R}^n$
  - $\pi_1(\mathbb{C} \setminus \{0\}) \cong \pi_1(S^1) = \mathbb{Z}$
  - $\pi_1(\mathbb{C} \setminus \{n \text{ points}\}) \cong \pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{n}) = F_n$ ← free group on n letters



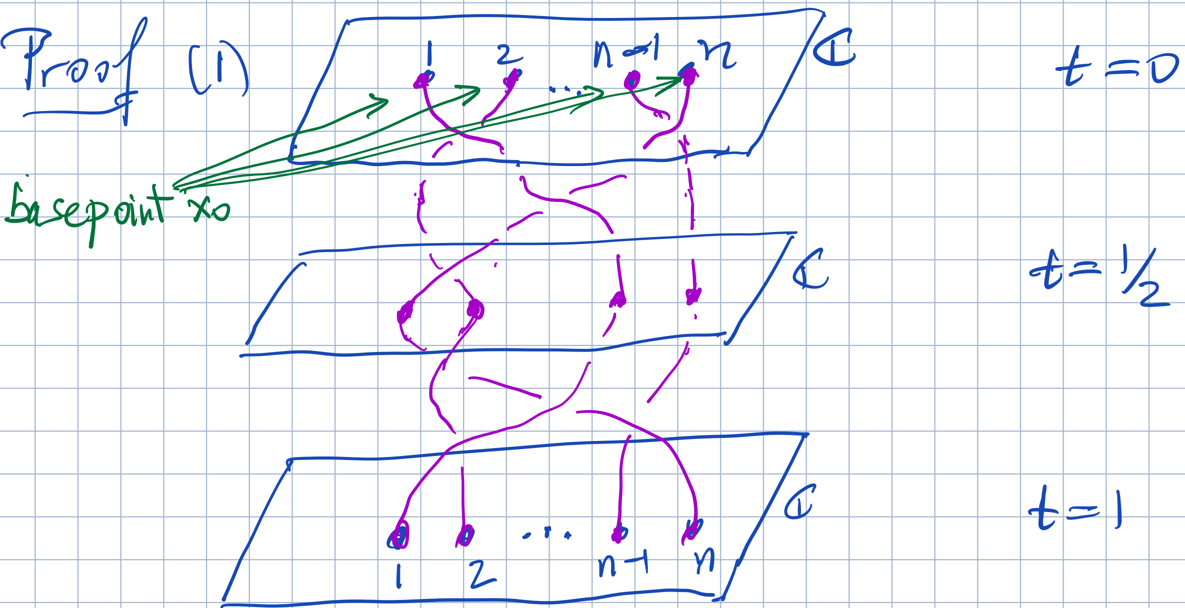


Theorem (Fox & Fadell – 1962)

(1) $\pi_1(\mathrm{Conf}_n(\mathbb{C})) \cong P_n$ — pure braid group on n strings

(2) $\pi_1(\mathrm{UConf}_n(\mathbb{C})) \cong B_n$ — braid group on n strings

Proof (1)



(2) similar

QED

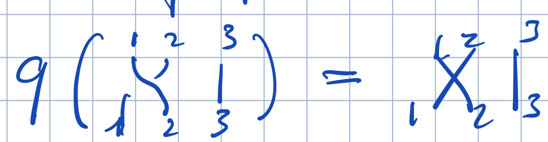
- Now recall the S_n-cover

$$\mathrm{Conf}_n(\mathbb{C}) \longrightarrow \mathrm{UConf}_n(\mathbb{C})$$

Using the relationship between covers and fundamental group, we get exact sequence

$$1 \longrightarrow P_n \xrightarrow{=\ker(q)} B_n \xrightarrow{q} S_n \longrightarrow 1$$

where q sends a braid β to the induced permutation of the strands, e.g.,



• Examples:

$n=2$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_2 & \longrightarrow & B_2 & \longrightarrow & S_2 & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \end{array}$$

$n=3$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_3 & \longrightarrow & B_3 & \longrightarrow & S_3 & \longrightarrow & 1 \\ & & \wr\| & & & & & & \\ & & \mathbb{Z}\times F_2 & & & & & & \end{array}$$

• Remark In turn, the braid groups completely determine the homotopy types of the respective configuration spaces:

$$\mathrm{Conf}_n(\mathbb{C}) \simeq K(P_n, 1)$$
$$U\mathrm{Conf}_n(\mathbb{C}) \simeq K(B_n, 1).$$

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9. Spaces of polynomials

• We may identify $\mathbb{C}^n$ with the space of all monic polynomials with coefficients in $\mathbb{C}$:

$$\boxed{\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\cong} & \mathrm{Poly}_n(\mathbb{C}) \\ (a_1, a_2, \dots, a_n) & \longrightarrow & P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n \end{array}}$$

• Vieta's formulas provide an identification

$$\mathbb{C}^n/S_n \xrightarrow[\text{Vieta's map}]{\cong} \mathbb{C}^n$$

$$(z_1, \dots, z_n) \longrightarrow (a_1, \dots, a_n)$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$P(x) = (x - z_1) \cdots (x - z_n) \qquad P(x) = x^n + a_1 x^{n-1} + \dots + a_n$$

- Some of the roots may be repeated. So let:

$$\boxed{SPoly_n(\mathbb{C}) = \left\{ \begin{array}{l} \text{space of polynomials} \\ \text{with no repeated linear factors} \end{array} \right\}}$$

or, the space of square-free polynomials.

- There is then an identification

$$SPoly_n(\mathbb{C}) \xrightarrow{\cong} UConf_n(\mathbb{C})$$

$$(x - z_1) \cdots (x - z_n) \longleftrightarrow (z_1, \dots, z_n)$$

- Therefore, $\pi_1(SPoly_n(\mathbb{C})) \cong B_n$.

- To conclude, let us describe in more concrete terms the space $SPoly_n(\mathbb{C})$

- Note that $P(x) = (x - z_1) \cdots (x - z_n)$ has a repeated root precisely when the polynomial

$$\Delta_n(z) := \prod_{1 \le i < j \le n} (z_i - z_j)^2$$

vanishes.

- This polynomial can be re-interpreted as a polynomial in the variables $a = (a_1, \dots, a_n)$

$$\Delta_n(a)$$

via the Vieta formulas.

- Therefore:

$$\boxed{SPoly_n(\mathbb{C}) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid \Delta_n(a) \neq 0\}}$$

i.e., the complement in $\mathbb{C}^n$ of the discriminant hypersurface $\Delta_n = 0$.

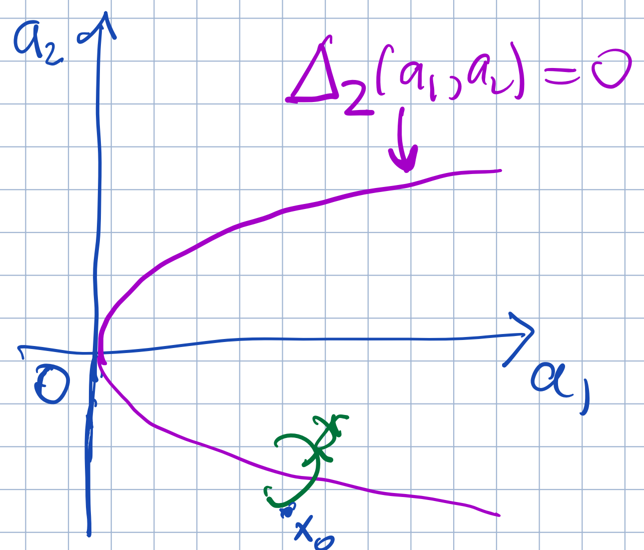
- Let us describe these hypersurfaces in low degrees ($n = 2, 3, 4$).

Quadrics

- Let $P(x) = (x - z_1)(x - z_2)$
$$= x^2 + a_1 x + a_2$$
where $a_1 = -(z_1 + z_2)$, $a_2 = z_1 z_2$

$\therefore\ \Delta_2 = (z_1 - z_2)^2 = (z_1 + z_2)^2 - 4 z_1 z_2$

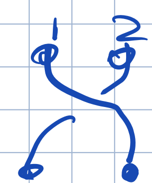
$$\therefore\ \boxed{\Delta_2(a) = a_1^2 - 4a_2}$$



$$B_2 = \pi_1\left(\mathbb{C}^2 \setminus \{\Delta_2 = 0\}\right) \cong \mathbb{Z}$$

generated by the loop γ around the parabola

That is: $\gamma \longleftrightarrow$



loop

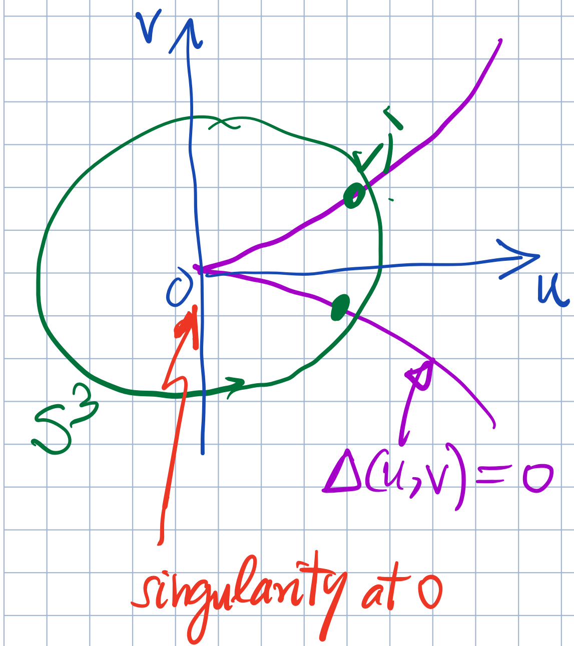
braid σ_1

Cubics

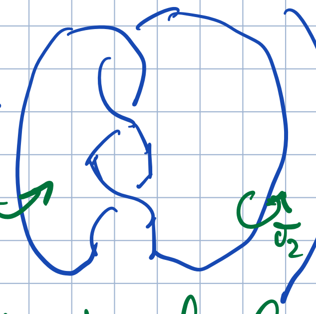
- Let $P(x) = x^3 + a_1 x + a_2 x^2 + a_3 x^3$
- Changing variables via $x = y - \frac{a_1}{3}$ replaces P by the depressed cubic
$$Q(y) = y^3 + uy + v$$

- Up to sign, then, the discriminant is

$$\Delta(u,v) = 4u^3 + 27v^2$$



$$B_3 = \pi_1\left(\mathbb{C}^3 \setminus \{\Delta_3 = 0\}\right) = \pi_1\left(\mathbb{C}^2 \setminus \{\Delta(u,v)=0\}\right) = \pi_1\left(S^3 \setminus K\right)$$



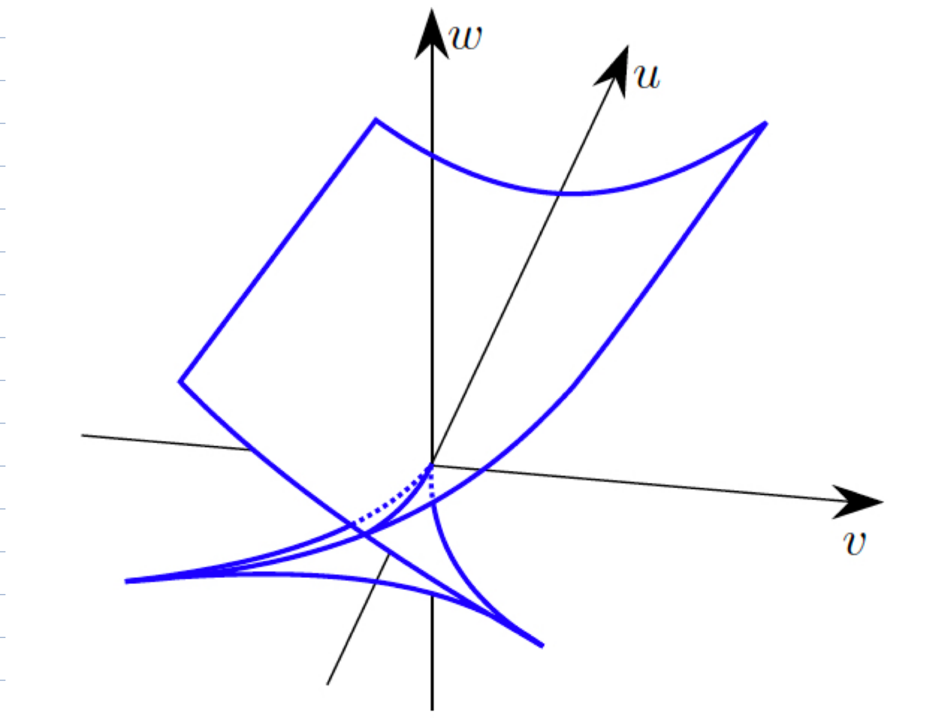
$$= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Quartics

- Let $P(x) = x^4 + ux^2 + vx + w$ be a depressed quartic
- The discriminant polynomial becomes

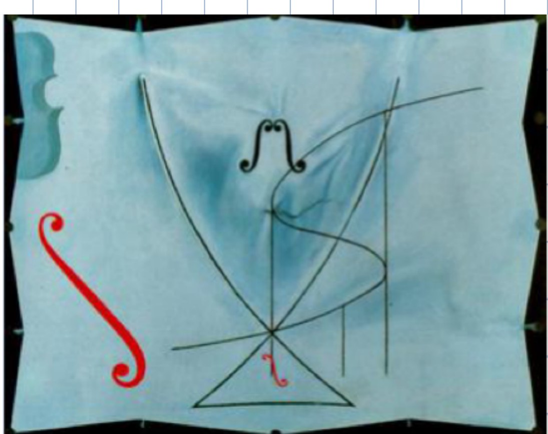
$$\Delta(u,v,w) = 8u^4 w - 4u^3v^2 - 128u^2w^2 + 144uv^2w - 27v^4 + 256w^3.$$

- The corresponding hypersurface (in 3-space) is called the <u>swallowtail</u> singularity.



after this bird:

- Thus: $B_4 = \pi_1\left(\mathbb{C}^3 \setminus \{\Delta(u,v,w)=0\}\right)$



The Swallow's Tail
– by Salvador Dalí
(1983)