

# HOMOTOPY CATEGORIES OF RINGS

Properties and consequences in module categories

Manuel Cortés Izordia



UNIVERSIDAD  
DE MÁLAGA

| [uma.es](http://uma.es)

# INTRODUCTION

$R$  = Noncommutative ring with unit

$\text{Mod-}R$  = The category of right modules.

$C(R)$  = Cocchain complexes with  $k$  chain maps.

$$\dots \rightarrow X^{n-1} \xrightarrow{d_x^{n-1}} X^n \xrightarrow{d_x^n} X^{n+1} \rightarrow \dots$$

$\mathcal{A}$  = Additive subcategory of  $\text{Mod-}R$

$K(\mathcal{A})$  = Homotopy category

- Objects: Cocchain complexes.

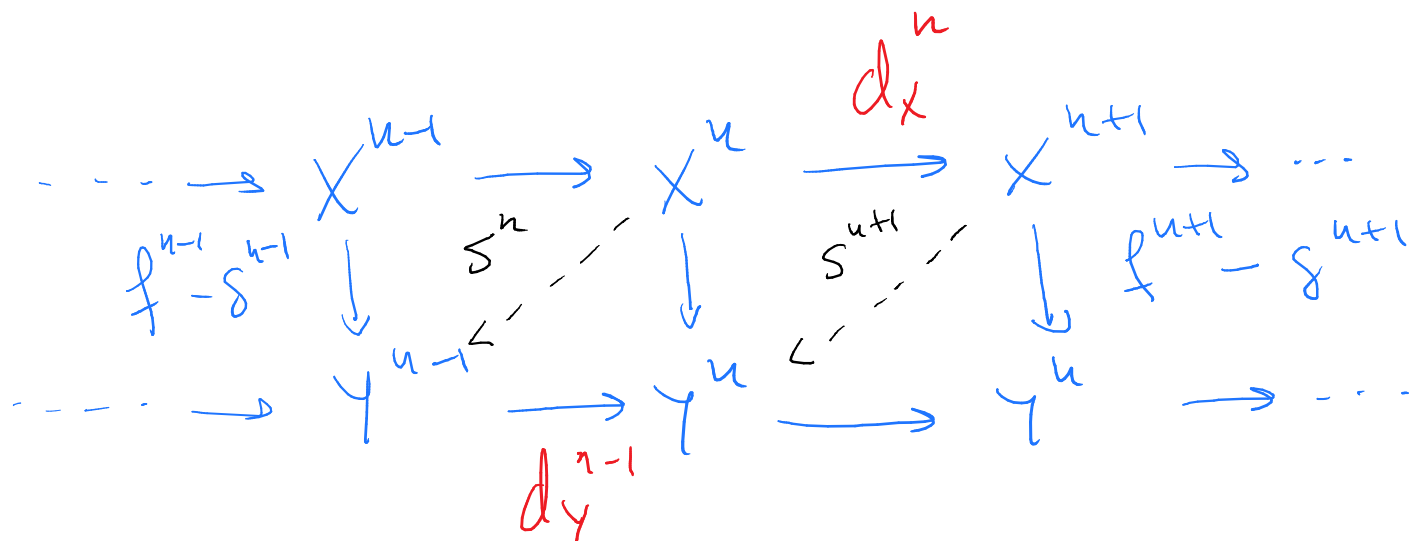
- Morphisms: homotopy equivalence classes of chain maps

# EQUIVALENT LOCHAIN MAPS

Given  $X, Y \in C(\mathbb{R})$  and  $f, g: X \rightarrow Y$

$$f \sim g \Leftrightarrow \exists s^n: X^n \rightarrow Y^{n-1} \text{ with}$$

$$f^n - g^n = d_y^{n-1} s^n + s^{n+1} d_x^n$$



# PROPERTIES OF $\mathbb{K}(A)$

①  $\mathbb{K}(A)$  is a triangulated category

a) Additive

b) There is an isomorphism

(suspension)  $\Sigma: \mathbb{K}(A) \rightarrow \mathbb{K}(A)$

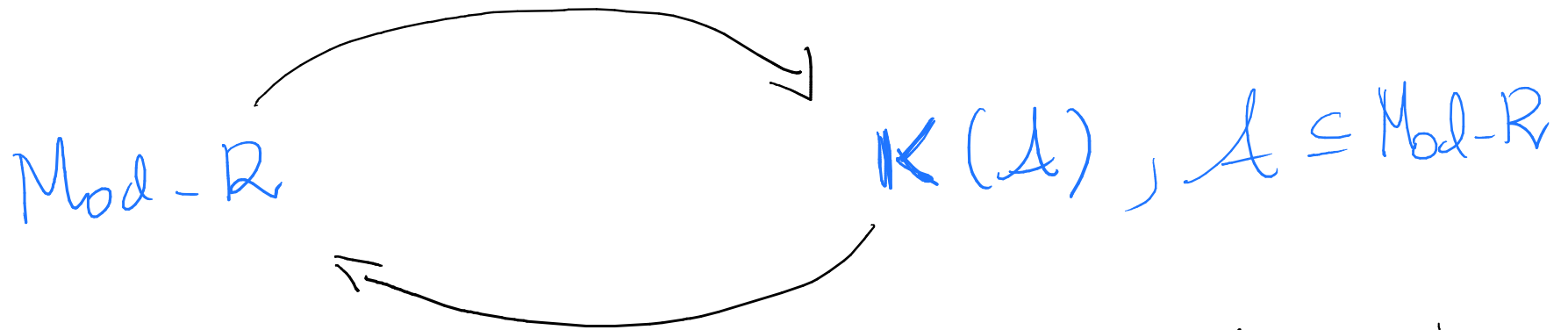
c) There is a class of composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

called triangles

# MAIN IDEA OF THE TALK

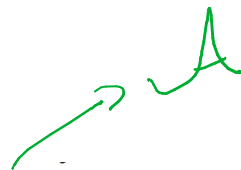
a) Study  $\mathbb{K}(A)$  using  
module theory



b) Study  $\mathbb{K}(A)$  using triangulated cat.  
and obtaining consequences in  $\text{Mod-}R$

# FROM MOD-R TO $K(A)$

Necuman, Inv. Math, 2008



**Theorem 1.1.** The homotopy category  $K(R\text{-Proj})$  is always  $\aleph_1$ -compactly generated, and as a consequence satisfies Brown representability. But it need not be compactly generated. Precisely

- (i) If  $R$  is right coherent then  $K(R\text{-Proj})$  is compactly generated.
- (ii) We give an example of an  $R$  for which  $K(R\text{-Proj})$  is not compactly generated.

$\aleph_1$ -compactly generated

$K(R\text{-Proj})$  has a set of generators  $\mathcal{S}$  such that every  $S \in \mathcal{S}$  is  $\aleph_1$ -compact:

For any  $f: S \rightarrow \bigoplus_{i \in I} k_i$ ,  $\exists J \subseteq I$  with  $|J| < \aleph_1$

$$S \rightarrow \bigoplus_{j \in J} k_j \xrightarrow{\neq} \bigoplus_{i \in I} k_i$$

## From Mod-R To $K(A)$

Saxén - Stovicek, Adv. Math, 2011

If  $A$  is a **deconstructible** class of modules, then  $K(A)$  is **coreflective** in  $K(\text{Mod-R})$  ( $K(A) \hookrightarrow K(\text{Mod-R})$  has a right adjoint)

$A$  is **deconstructible** if

exists a set of objects  $S^0 \subseteq A$  such that  $\forall A \in A$

- $A_\alpha \subseteq A_{\alpha+1}$

- $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ ,  $\beta$  limit

- $A_{\alpha+1}/A_\alpha \in S^0$

$$A = \bigcup_{\alpha < \kappa} A_\alpha$$

ordinal  $\leftarrow$

Then  $(A_\alpha \mid \alpha < \kappa)$  is the  $S^0$ -filtration of  $A$

# DECONSTRUCTIBLE CLASSES

## Example 1: Finite length modules

$M$  has finite length if it contains a chain of submodules

$$0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots \subseteq M_n = M$$

such that there is no submodule between them

$$\iff \frac{M_{k+1}}{M_k} \text{ is simple}$$

$M$  has a finite  $S$ -filtration for

$S =$  The set of all simple modules



# DECONSTRUCTIBLE CLASSES

## Example 2: Vector spaces

If  $R = k$  is a field and  $V$  is a vector space.

- Take a basis  $\{v_\alpha \mid \alpha < \kappa\}$  ( $\kappa$  infinite,  $\kappa$  a cardinal)
- If  $V_\alpha = \langle v_\gamma \mid \gamma < \alpha \rangle$ , then  $(V_\alpha \mid \alpha < \kappa)$  is a filtration of  $V$  with

$$\dim_k \frac{V_{\alpha+1}}{V_\alpha} = 1.$$

Every vector space is filtered by 1-dimensional vector spaces

# FROM $\mathbf{K}(A)$ TO MOD- $R$

Acyclic

Neeman, Inv. Math, 2008

If we take a complex

$$\cdots \rightarrow P^{n-1} \xrightarrow{d^{n-1}} P^n \xrightarrow{d^n} P^{n+1} \rightarrow \cdots$$

$\Rightarrow \text{Ker } d^n \text{ is projective.}$

- $\text{Im } d^{n-1} = \text{Ker } d^n$

- $P^n$  projective

- $\text{Ker } d^n$  flat

Direct limit of projectives.

*Remark 2.15.* To illustrate the non-triviality of the implication (iii)  $\implies$  (i) let us note a curious aside. Suppose  $X$  is an acyclic chain complex

$$\cdots \xrightarrow{\partial^{i-2}} X^{i-1} \xrightarrow{\partial^{i-1}} X^i \xrightarrow{\partial^i} X^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

of projective modules. Suppose that, for each  $i \in \mathbb{Z}$ , the image  $I^i$  of the differential  $\partial^i : X^i \rightarrow X^{i+1}$  is a flat  $R$ -module.

By definition  $X$  belongs to  $\mathbf{K}(R\text{-Proj})$ , and by (iii)  $\implies$  (i)  $X$  also belongs to  $\mathbf{K}(R\text{-Proj})^\perp$ . Hence  $X$  must be null homotopic. The module  $I^i$ , being a direct summand of  $X^i$ , is forced to be projective. What is curious about this aside is that the statement is the assertion that certain flat modules have to be projective; it does not mention triangulated categories. I do not know an elementary proof, a proof which avoids homotopy categories.

## PERIODIC MODULES

Benson, Goodenough, Pac. J. Math, 2000

In an exact sequence in  $\text{Mod-}R$ ,

$$0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$

$f$  monic  
 $g$  epic  
 $\text{Im} f = \text{Ker} g$

$M$  projective,  $N$  flet  $\Rightarrow N$  projective

### Remark

We can form the following exact co-ex

$$\begin{array}{ccccccc} \dots & \rightarrow & M & \longrightarrow & M & \longrightarrow & M & \longrightarrow & \dots \\ & & \searrow & & \searrow & & \searrow & & \\ & & N & \xrightarrow{f} & N & \xrightarrow{f} & N & \xrightarrow{f} & \dots \end{array}$$

$$\text{ker } f = \text{Im } g \cong N \text{ flet}$$

# PERIODIC MODULES

## $\Lambda$ -periodic modules

Modules  $M$  appearing in a short exact sequence

$$0 \rightarrow M \rightarrow A \rightarrow M \rightarrow 0 \quad \text{with } A \in \mathcal{A}$$

## Main problem

When  $M$  belongs to  $\mathcal{A}$  as well?

Bazzoui, CI, Estrade, Alg. Rep. The., 2020

If  $\text{Cot} =$  class of cotorsion modules then

Every  $\text{Cot}$ -periodic module is cotorsion.

# FROM $K(A)$ TO MOD- $R$

Totally acyclic complexes  $X$

$$\cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots$$

- $X^n$  is projective
- Acyclic:  $\text{Ker } d^n = \text{Im } d^{n-1}$
- Totally acyclic

$$\begin{array}{ccc}
 X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} \\
 \downarrow f & & \swarrow g & & \\
 & & & & 
 \end{array}$$

$gd^n = f$

$P \in \text{Proj}, f d^{n-1} = 0$

Gorenstein projective modules

If  $M \cong \text{Ker } d^n$  for some totally acyclic complex  $X$ .

$$\text{Proj} \subseteq \text{GProj}$$

# GORENSTEIN HOMOLOGICAL ALGEBRA

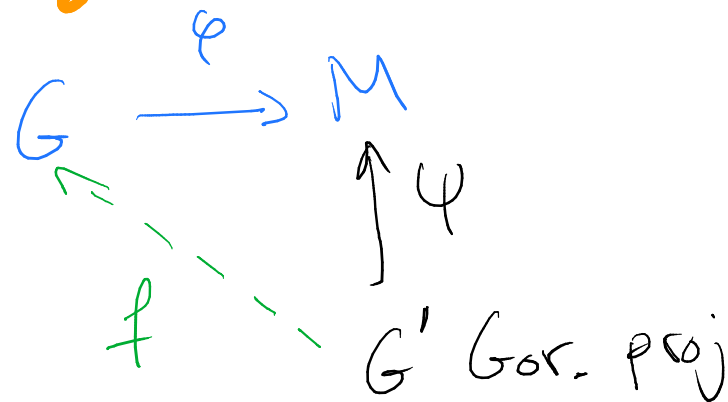
## Main open problem

Does there exist, for any module  $M$ , a morphism

$\varphi: G \rightarrow M$  with

- $G$  Gorenstein projective

- **Lifting property**



Gorenstein projective  
precovers

Good morphisms to make resolutions and  
"relative homological algebra"

## Theorem (CI-Sasoch, 2022)

Set theory hypothesis

$\forall$  cardinal  $\kappa$ ,  $\exists \lambda, \mu \geq \kappa$  such that  $\lambda$  is  $\text{Sym}$ -compact

$\implies$  Every module has a GP-precover for any ring.

## FROM $K(A)$ TO $\text{MOD-}R$

Jørgensen, J. Eur. Math. Soc. 2007

If  $\underline{\mathcal{X}}$  is coreflective in  $K(\text{Proj})$ , then every module has a Gorenstein-Projective precover.

Subcategory of  $C(\text{Mod-}R)$

$\mathcal{X}$  = All totally acyclic complexes.

Subcategory of  $K(\text{Proj})$

$\underline{\mathcal{X}}$  = The subcategory of  $K(\text{Proj})$  whose class of objects is  $\mathcal{X}$ .



# SUMMARIZING

## Module category

Facts on flat modules,  
countably generated, etc.

Neeman

$\Rightarrow$

Jacoin-Stovicek

## Homology category

$K(\text{Proj})$  is  $\mathcal{X}_1$ -compactly  
generated

$A$  is deconstructible

$\Rightarrow$

$K(A)$  is coreflective

Every flat and Proj-periodic  
module is projective

Neeman

$\Leftarrow$

Some results in  $K(\text{Proj})$

Every module has a  
GP-precover.

$\Leftarrow$

Jørgensen

$\mathcal{X}$  is coreflective

## OBJECTIVE OF THE TALK

Show recent results in this flavour

- a) Study homotopy categories of  $N$ -complexes  
CI-Torrecillas, BHMS, 2023
- b) Give new conditions that implies  $\underline{\mathcal{X}}$  is  
coreflective in  $K(\text{Proj})$  CI, Sci. Chin. Math, 2023



Every module has a GP-precover by  
Jørgensen's result.

## 2. HOMOTOPY CATEGORIES OF $N$ -COMPLEXES

Fix  $N \in \mathbb{N}$ ,  $N \geq 2$

Category of  $N$ -complexes  $X : C_N(\mathbb{R})$

- Objects:  $\cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots$
- Morphisms: Cochain maps.

$$d^n d^{n-1} \cdots d^{n-N+1} = 0$$

Homotopy category of an additive subcategory  $\mathcal{A} : K_N(\mathcal{A})$

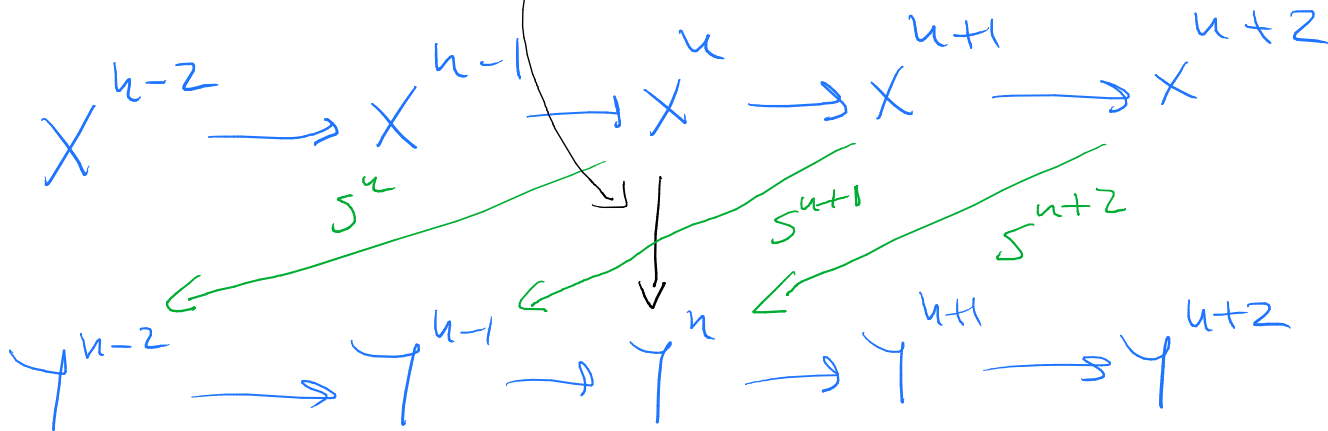
- Objects:  $N$ -complexes in  $\mathcal{A}$
- Morphisms: Homotopy equivalence classes

# HOMOLOGY EQUIVALENCE

If  $X, Y \in C_N(\mathbb{R})$  and  $f, g: X \rightarrow Y$  then  
 $f \sim g \iff \exists s^u: X^u \rightarrow Y^{u-N+1}$  such that

$$f^u - g^u = \sum_{j=0}^{N-1} d_Y^{\{j\}} s^{i+N-j-1} d_X^{\{N-j-1\}}$$

Case  $N=3$



$$\begin{aligned}
 f^u - g^u &= \\
 &= d^{u-1} d^u s^u \\
 &+ d^{k-1} s^{k+1} d^k \\
 &+ s^{k+2} d^{k+1} d^k
 \end{aligned}$$

## 2.1. COREFLECTIVE SUBCATEGORIES OF $K_N(\text{Mod-}R)$

**QUESTION**: How can we construct coreflective subcategories of  $K_N(\text{Mod-}R)$ ?

Following Saorin - Stovicek

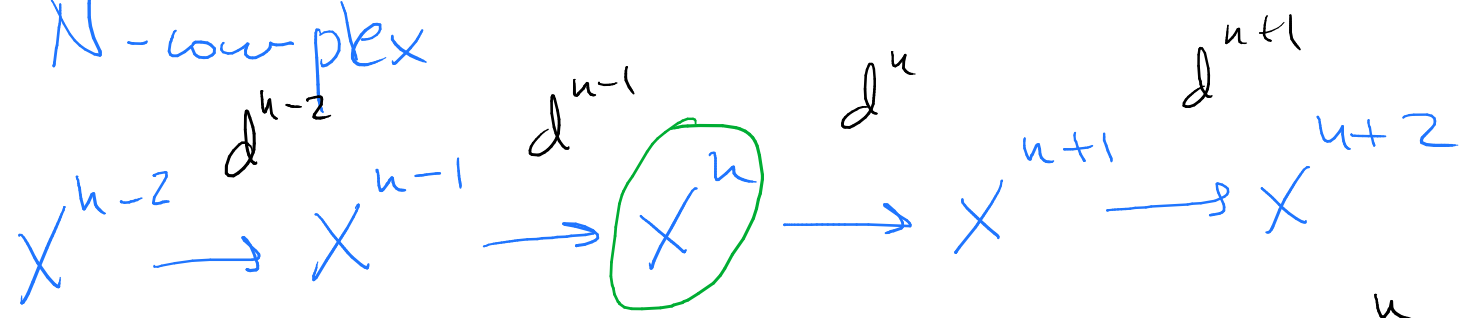
- We take a deconstructible class  $\mathcal{A} \subseteq \text{Mod-}R$
- Construct subcategories of  $K(\text{Mod-}R)$ 
  - a)  $k(\mathcal{A})$  (Remember:  $N$ -complexes from  $\mathcal{A}$ )
  - b)  $\underline{E}(\mathcal{A})$ :  $N$ -acyclic complex from  $\mathcal{A}$

# N-ACYCLIC COMPLEXES

$X$  is  $N$ -acyclic if all  $N$ -homology modules vanish.

## Case $N=3$

Given an  $N$ -complex



$$d^n d^{n-1} d^{n-2} = 0 \implies \text{Im } d^{n-2} d^{n-1} \subseteq \text{Ker } d^n$$

$$d^{n+1} d^n d^{n-1} = 0 \implies \text{Im } d^{n-1} \subseteq \text{Ker } d^{n+1} d^n$$

$X$  is  $N$ -acyclic if  $\text{Im } d^{n-2} d^{n-1} = \text{Ker } d^n$  and  $\text{Im } d^{n-1} = \text{Ker } d^{n+1} d^n$

## 2.1. COREFLECTIVE SUBCATEGORIES

### Theorem

If  $\mathcal{A} \subseteq \text{Mod-}R$  is deconstructible  $\Rightarrow \mathbb{K}(\mathcal{A})$  and  $\underline{E}(\mathcal{A})$  are coreflective subcategories of  $\mathbb{K}(\text{Mod-}R)$

### Proof

$\mathcal{A}$  deconstructible  $\stackrel{\textcircled{1}}{\Rightarrow}$   
 $\stackrel{\textcircled{1}}{\Rightarrow} C_{\mathcal{N}}(\mathcal{A})$  and  $E_{\mathcal{N}}(\mathcal{A})$  are deconstructible in  $C_{\mathcal{N}}(R)$   
 $\stackrel{\textcircled{2}}{\Rightarrow} C(\mathcal{A})$  and  $E(\mathcal{A})$  are precovering in  $C(R)$   
 $\stackrel{\textcircled{3}}{\Rightarrow} \mathbb{K}(\mathcal{A})$  and  $\underline{E}(\mathcal{A})$  are coreflective in  $\mathbb{K}(\text{Mod-}R)$  ~~///~~

## 2.1. COREFLECTIVE SUBCATEGORIES

①  $C(\mathcal{A})$  and  $E(\mathcal{A})$  are deconstructible.

If  $X \in C(\mathcal{A})$ , we have to find some filtration of  $X$

$$(X_\alpha \mid \alpha \leq \kappa)$$

$$\begin{array}{ccccccc}
 X_0: & \cdots & \rightarrow & X_0^{n-1} & \rightarrow & X_0^n & \rightarrow & X_0^{n+1} & \rightarrow & \cdots \\
 \downarrow & & & \downarrow^{n-1} & & \downarrow^n & & \downarrow^{n+1} & & \\
 X_1: & \cdots & \rightarrow & X_1^{n-1} & \rightarrow & X_1^n & \rightarrow & X_1^{n+1} & \rightarrow & \cdots \\
 \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
 \frac{X_1}{X_0} & \cdots & \rightarrow & \frac{X_1^{n-1}}{X_0^{n-1}} & \rightarrow & \frac{X_1^n}{X_0^n} & \rightarrow & \frac{X_1^{n+1}}{X_0^{n+1}} & \rightarrow & \cdots
 \end{array}$$

Deconstruction  
Hill's lemma

→ Belonging to some set of complexes.



## 2.1. COREFLECTIVE SUBCATEGORIES

(2)  $C_X(\mathcal{A})$  and  $E_X(\mathcal{A})$  are precovering

**Theorem** (Enochs, Saorin-Stovicek)

Every deconstructible class is precovering

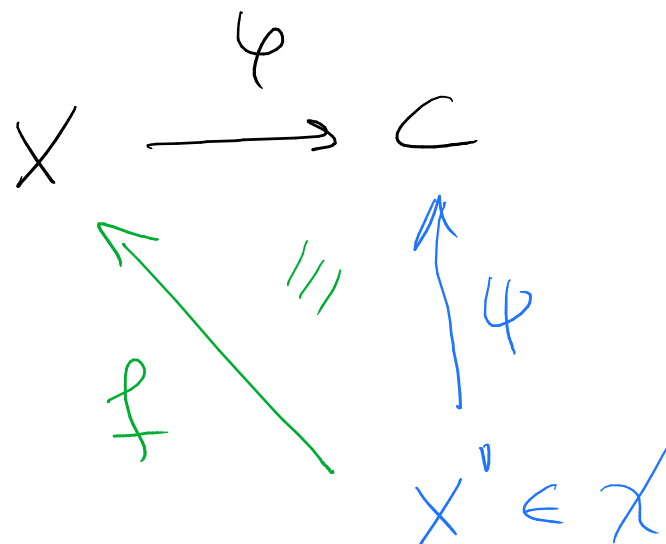
**Precovering class  $\mathcal{X}$  in a category  $\mathcal{C}$**

For every  $C \in \mathcal{C}$  exists a morphism  $\varphi: X \rightarrow C$  with

a)  $X \in \mathcal{X}$

b) **Lifting property**

$$\varphi f = \psi$$



## 2.1. COREFLECTIVE SUBCATEGORIES

③  $K_N(A)$  and  $E_N(A)$  are coreflective in  $K_N(\text{Mod-}R)$

③.1  $K_N(A)$  and  $E_N(A)$  are precovering in  $K_N(\text{Mod-}R)$

Theorem (Cortés-Izurdiaga-Crivei-Saorín, 2022)

If  $\mathcal{A} \subseteq \mathcal{C}$  is a subcategory of an additive category with split idempotents, then  $\mathcal{A}$  is coreflective  $\iff$

a)  $\mathcal{A}$  is precovering.

b)  $\mathcal{A}$  is closed under direct summands.

c) Every morphism in  $\mathcal{A}$  has a pseudocokernel in  $\mathcal{C}$  which belongs to  $\mathcal{A}$ .

## 2.2. $K_n(\text{Proj})$ IS $\chi_1$ -COMPACTLY GENERATED

### Theorem

$K_n(\text{Proj})$  is  $\chi_1$ -compactly generated.

### Proof

The proof uses deconstruction and fill lemma but with some particularities.

↓ a)  $\text{Proj}$  is more than deconstructible: decomposable.

### Kaplansky's Theorem

$P \in \text{Proj} \Rightarrow P = \bigoplus_{i=1}^{\infty} P_i$  with  $P_i$  compactly generated

## 2.2. $K_N(\text{Proj})$ IS $\aleph_1$ -COMPACTLY GENERATED

b)  $C_N(\text{Proj})$  is more than *deconstructible*

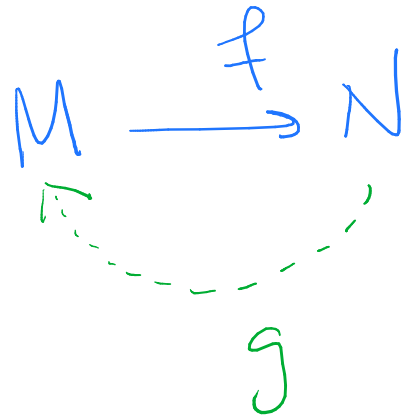
If  $X \in C_N(\text{Proj})$ , then the filtration  $(X_\alpha \mid \alpha < \kappa)$

$$\begin{array}{ccccccc}
 X_0: & \dots & \rightarrow & X_0^{n-1} & \rightarrow & X_0^n & \rightarrow & X_0^{n+1} & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \text{split mono.} \\
 & & & X_1^{n-1} & \rightarrow & X_1^n & \rightarrow & X_1^{n+1} & \rightarrow & \dots \\
 X_1: & \dots & \rightarrow & X_1^{n-1} & \rightarrow & X_1^n & \rightarrow & X_1^{n+1} & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \text{split epi.} \\
 & & & X_1^{n-1} & \rightarrow & X_1^n & \rightarrow & X_1^{n+1} & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & X_1 & \rightarrow & X_1 & \rightarrow & X_1 & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & X_0^{n-1} & \rightarrow & X_0^n & \rightarrow & X_0^{n+1} & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & X_1 & \rightarrow & X_1 & \rightarrow & X_1 & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & X_0 & \rightarrow & X_0 & \rightarrow & X_0 & \rightarrow & \dots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & X_0 & \rightarrow & X_0 & \rightarrow & X_0 & \rightarrow & \dots
 \end{array}$$

$\hookrightarrow$  countably generated projective

# SPLITTING MORPHISMS

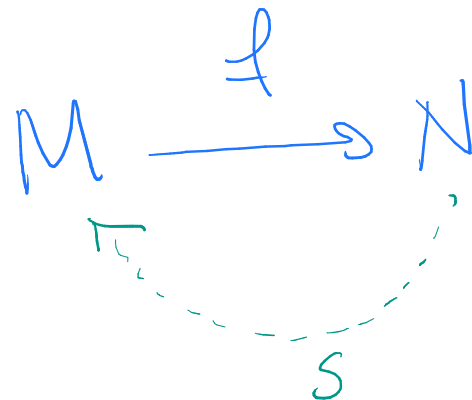
## SPLIT MONOMORPHISM



$$gf(x) = x, \forall x \in M$$

$$gf = 1_M$$

## SPLIT EPIMORPHISM



$$fg(x) = x, \forall x \in N$$

$$fg = 1_N$$

### 3. EXISTENCE OF GP-PRELOVERS

- We work in  $K(\text{Proj})$  and consider  $K = \underline{\mathcal{X}} \subseteq K(\text{Proj})$  where  $\mathcal{X}$  are the **totally acyclic complexes**.

**Remember**

$$\cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots$$

- $X^n$  is projective
- Acyclic**:  $\text{Ker } d^n = \text{Im } d^{n-1}$
- Totally acyclic**

$$\begin{array}{ccccc} X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} \\ & & \downarrow f & \swarrow g & \\ & & P \in \text{Proj} & & \end{array} \quad gd^n = f$$

$$P \in \text{Proj}, f d^n = 0$$

### 3. EXISTENCE OF GP-PRELOVERS

#### Class of morphisms $\text{Mor}_K$

- Take a morphism  $f: X \rightarrow Y$  in  $\mathbb{K}(\text{Proj})$
- By the axioms of triangulated categories

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X$$

- Then  $f \in \text{Mor}_K \iff Z \in K$

#### Relevance of $\text{Mor}_K$

It allows to define the Verdier quotient

$$\mathbb{K}(\text{Proj})/K$$

### 3. EXISTENCE OF GP-PRECOVERS

#### Theorem

If  $\text{Mor}_K$  satisfies the generalized Baer lemma in  $\mathbb{K}(\text{Proj})$ , then there exists GP-precovers in  $\text{Mod-}R$ .

#### $\text{Mor}_K$ -injective objects

$X \in \mathbb{K}(\text{Proj})$  is  $\text{Mor}_K$ -injective if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \in \text{Mor}_K \\ \downarrow g & \dashrightarrow h & \\ X & & \end{array}$$

$$hf = g$$



### 3. EXISTENCE OF GP-PRELOVERS

#### Generalized Baer Lemma

$\text{Mod}_R$  satisfies the generalized Baer lemma if there is a set  $N \subseteq \text{Mod}_R$  (not a class!) such that

$X$  is  $\text{Mod}_R$ -injective  $\iff X$  is  $N$ -injective

#### Example

$M \in \text{Mod-}R$  is injective  $\iff M$  is injective w.r.t. any monomorphism

• Classical Baer lemma

$M$  is injective  $\iff M$  is  $N$ -injective

$N$   
"  
Monomorphisms  
 $f: I \rightarrow R$

### 3. EXISTENCE OF GP-PRELOVERS

Theorem (CI-Guil-Kaleboğaz-Srivastava, 2020)

If  $(\mathcal{C}; \mathcal{E})$  is an exact category satisfying certain conditions such that the class of inflations satisfy the Generalized Baer lemma, then

$(\mathcal{C}; \mathcal{E})$  has enough injectives.

### 3. EXISTENCE OF GP-PRELOVERS

#### Theorem

If  $\text{Mod}_K$  satisfies the generalized Baer lemma in  $K(\text{Proj})$ , then there exists GP-precovers in  $\text{Mod-R}$ .

#### Proof

a) The Verdier quotient  $K(\text{Proj})/K$  has small local-sets.

b)  $K$  is coreflective in  $K(\text{Proj})$

Using results from triangulated categories

c) There exist GP-precovers

By Jorgensen result

✘

### 3. EXISTENCE OF GP-PRELOVERS

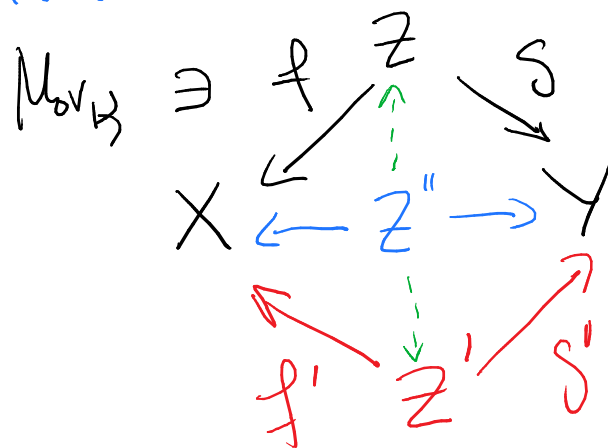
a) The Verdier quotient  $\mathbb{K}(\text{Proj})/K$  has small hom-sets.

- The category  $\mathbb{K}(\text{Proj})/K$

- Objects:  $\mathbb{K}(\text{Proj})$

- Morphisms between  $X$  and  $Y$

- Morphisms between  $X$  and  $Y$   
Equivalent classes of triples  $(Z, f, s)$



It need not be a set!

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