HOMOTOPY CATEGORIES OF RINGS

Properties and consequences in model categories

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INTRODUCTION

R = Noncommutative ring with unit

Mod-R = The category of right modules.

$C(R)$ = Cochain complexes with chain maps.

$$\cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to$$

$\mathcal{A}$ = Additive subcategory of Mod-R

$K(\mathcal{A})$ = Homotopy category

- <u>Objects</u>: Cochain complexes.
- <u>Morphisms</u>: homotopy equivalence classes of cochain maps

EQUIVALENT COCHAIN MAPS

Given $X, Y \in C(R)$ and $f, g : X \longrightarrow Y$

$$f \sim g \iff \exists\, s^n : X^n \longrightarrow Y^{n-1} \text{ with}$$
$$f^n - g^n = d_Y^{n-1} s^n + s^{n+1} d_X^n$$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "f^{n-1}-g^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d] \arrow[dl, dashed, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}-g^{n+1}"] \arrow[dl, dashed, "s^{n+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r] & Y^n \arrow[r] & \cdots
\end{tikzcd}$$

PROPERTIES OF $\mathbb{K}(\mathcal{A})$

① $\mathbb{K}(\mathcal{A})$ is a triangulated category

a) Additive

b) There is an isomorphism

(suspension) $\Sigma : \mathbb{K}(\mathcal{A}) \longrightarrow \mathbb{K}(\mathcal{A})$

c) There is a class of composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

called triangles

MAIN IDEA OF THE TALK

a) Study $\mathbb{K}(\mathcal{A})$ using module theory

$$\begin{tikzcd} \text{Mod-}R \arrow[r, bend left] & \mathbb{K}(\mathcal{A}) \arrow[l, bend left] \end{tikzcd} \quad , \ \mathcal{A} \subseteq \text{Mod-}R$$

b) Study $\mathbb{K}(\mathcal{A})$ using triangulated cat. and obtaining consequences in Mod-R

FROM MOD-R TO K(A)

Neeman, Inv. Math, 2008

Theorem 1.1. *The homotopy category* **K**(R-Proj) *is always* $\aleph_1$*-compactly generated, and as a consequence satisfies Brown representability. But it need not be compactly generated. Precisely*

(i) *If R is right coherent then* **K**(R-Proj) *is compactly generated.*

(ii) *We give an example of an R for which* **K**(R-Proj) is not *compactly generated.*

A

$\aleph_1$-compactly generated

$K(R\text{-Proj})$ has a set of generators $\mathcal{S}$ such that every $S \in \mathcal{S}$ is $\aleph_1$-compact:

For any $f : S \longrightarrow \bigoplus_{i \in I} K_i$, $\exists\ J \subseteq I$ with $|J| < \aleph_1$

$$\begin{tikzcd} S \arrow[r] \arrow[rr, bend left, "f"] & \bigoplus_{j \in J} K_j \arrow[r, hook] & \bigoplus_{i \in I} K_i \end{tikzcd}$$

From Mod-R to $\mathbb{K}(\mathcal{A})$

Saorín – Šťovíček, Adv. Math, 2011

If $\mathcal{A}$ is a deconstructible class of modules, then $\mathbb{K}(\mathcal{A})$ is coreflective in $\mathbb{K}(\text{Mod-}R)$ ($\mathbb{K}(\mathcal{A}) \hookrightarrow \mathbb{K}(\text{Mod-}R)$ has a right adjoint)

$\mathcal{A}$ is deconstructible if

exists a set of objects $\mathcal{S} \subseteq \mathcal{A}$ such that $\forall A \in \mathcal{A}$

$$A = \bigcup_{\alpha < \kappa} A_\alpha$$

ordinal

- $A_\alpha \subseteq A_{\alpha+1}$
- $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$, β limit
- $A_{\alpha+1}/A_\alpha \in \mathcal{S}$

Then $(A_\alpha \mid \alpha < \kappa)$ is the $\mathcal{S}$-filtration of A

DECONSTRUCTIBLE CLASSES

Example 1: Finite length modules

M has finite length if it contains a chain of submodules

$$0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_n = M$$

such that there is no submodule between them

$$\Longleftrightarrow \frac{M_{k+1}}{M_k} \text{ is simple}$$

M has a finite $\mathcal{S}$-filtration for

$\mathcal{S}$ = The set of all simple modules

DECONSTRUCTIBLE CLASSES

Example 2: Vector spaces

If $R = k$ is a field and V is a vector space.

- Take a basis $\{v_\alpha \mid \alpha < \kappa\}$ (infinite, κ a cardinal)
- If $V_\alpha = \langle v_\gamma \mid \gamma < \alpha \rangle$, then $(V_\alpha \mid \alpha < \kappa)$ is a filtration of V with

$$\dim_k \frac{V_{\alpha+1}}{V_\alpha} = 1.$$

Every vector space is filtered by 1-dimensional vector spaces

FROM K(𝒜) TO MOD-R

Neeman, Inv. Math, 2008

If we take a complex

$$\cdots \longrightarrow P^{n-1} \xrightarrow{d^{n-1}} P^{n} \xrightarrow{d^{n}} P^{n+1} \longrightarrow \cdots$$

Acyclic

- $\operatorname{Im} d^{n-1} = \ker d^{n}$
- P^{n} projective
- $\ker d^{n}$ flat

Direct limit of projectives.

$\Longrightarrow$ $\ker d^{n}$ is projective.

Remark 2.15. To illustrate the non-triviality of the implication (iii) $\Longrightarrow$ (i) let us note a curious aside. Suppose X is an acyclic chain complex

$$\xrightarrow{\partial^{i-2}} X^{i-1} \xrightarrow{\partial^{i-1}} X^{i} \xrightarrow{\partial^{i}} X^{i+1} \xrightarrow{\partial^{i+1}}$$

of projective modules. Suppose that, for each $i \in \mathbb{Z}$, the image I^i of the differential $\partial^i : X^i \longrightarrow X^{i+1}$ is a flat R-module.

By definition X belongs to $\mathbf{K}(R\text{-Proj})$, and by (iii) $\Longrightarrow$ (i) X also belongs to $\mathbf{K}(R\text{-Proj})^{\perp}$. Hence X must be null homotopic. The module I^i, being a direct summand of X^i, is forced to be projective. What is curious about this aside is that the statement is the assertion that certain flat modules have to be projective; it does not mention triangulated categories. I do not know an elementary proof, a proof which avoids homotopy categories.

PERIODIC MODULES

Benson, Goodearl, Pac. J. Math, 2000

In an exact sequence in Mod-R,

$$0 \to N \xrightarrow{f} M \xrightarrow{g} N \to 0 \quad \left\{ \begin{array}{l} f \text{ monic} \\ g \text{ epic} \\ \operatorname{Im} f = \ker g \end{array} \right.$$

M projective, N flat $\Rightarrow$ N projective

Remark

We can form the following exact complex

$$\begin{tikzcd}
\cdots \arrow[r] & M \arrow[rr] \arrow[dr, "g"'] & & M \arrow[rr] \arrow[dr, "g"'] & & M \arrow[r] & \cdots \\
& & N \arrow[ur, "f"'] & & N \arrow[ur, "f"'] & &
\end{tikzcd}$$

$\ker f = \operatorname{Im} g \cong N$ flat

PERIODIC MODULES

$\mathcal{A}$-periodic modules

Modules M appearing in a short exact sequence

$$0 \to M \to A \to M \to 0 \qquad \text{with } A \in \mathcal{A}$$

Main problem

When M belongs to $\mathcal{A}$ as well?

Bazzoni, CI, Estrada, Alg. Rep. The., 2020

If Cot = Class of cotorsion modules then

Every Cot-periodic module is cotorsion.

From $K(\mathcal{A})$ to Mod-R

Totally acyclic complexes X

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots$$

- X^n is projective
- Acyclic: $\operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$
- Totally acyclic

$$\begin{tikzcd} X^{n-1} \arrow[r, "d^{n-1}"] & X^n \arrow[r, "d^n"] \arrow[d, "f"'] & X^{n+1} \arrow[dl, dashed, "g"] \\ & P & \end{tikzcd}$$

$\exists\, g$ with $gd^n = f$

$P \in \mathrm{Proj}$, $fd^{n-1} = 0$

Gorenstein projective modules

If $M \cong \operatorname{Ker} d^n$ for some totally acyclic complex X.

$$\boxed{\mathrm{Proj} \subseteq \mathrm{GProj}}$$

GORENSTEIN HOMOLOGICAL ALGEBRA

Main open problem

Does there exists, for any module M, a morphism

$\varphi: G \to M$ with
- G Gorenstein projective
- **Lifting property**

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & M \\
& G' \text{ Gor. proj} \arrow[u, "\psi"'] \arrow[ul, dashed, "f"]
\end{tikzcd}$$

Gorenstein projective precovers

$\downarrow$

Good morphisms to make resolutions and "relative homological algebra"

Theorem (CI – Saroch, 2022)

Set theory hypothesis

$\forall$ cardinal κ, $\exists\ \lambda, \mu \geq \kappa$ such that λ is $\mathcal{L}_{\mu\omega}$-compact

$\Longrightarrow$ Every module has a GP-precover for any ring.

FROM K(A) TO MOD-R

Jørgensen, J. Eur. Math. Soc. 2007

If $\underline{\mathcal{X}}$ is coreflective in $\mathbb{K}(\mathrm{Proj})$, then every module has a Gorenstein-Projective precover.

Subcategory of $C(\text{Mod-}R)$

$\mathcal{X}$ = All totally acyclic complexes.

Subcategory of $K(\mathrm{Proj})$

$\underline{\mathcal{X}}$ = The subcategory of $K(\mathrm{Proj})$ whose class of objects is $\mathcal{X}$.

SUMMARIZING

Module category		Homotopy category
Facts on flat modules, countably generated, etc.	$\xRightarrow{\text{Neeman}}$	$K(\mathrm{Proj})$ is $\aleph_1$-compactly generated
$\mathcal{A}$ is deconstructible	$\xRightarrow{\text{Šaorín–Šťovíček}}$	$K(\mathcal{A})$ is coreflective
Every flat and Proj-periodic module is projective	$\xLeftarrow{\text{Neeman}}$	Some results in $K(\mathrm{Proj})$
Every module has a GP-precover.	$\xLeftarrow{\text{Jørgensen}}$	$\underline{\mathcal{X}}$ is coreflective

OBJECTIVE OF THE TALK

Show recent results in this flavour

a) Study homotopy categories of N-complexes
CI-Torrecillas, BMMS, 2023

b) Give new conditions that implies $\underline{\mathcal{X}}$ is coreflective in $\mathbb{K}(\mathrm{Proj})$ CI, Sci. Chin. Math, 2023

$\Downarrow$

Every module has a GP-precover by Jørgensen's result.

2. HOMOTOPY CATEGORIES OF N-COMPLEXES

Fix $N \in \mathbb{N}$, $N \geq 2$

Category of N-complexes X : $C_N(R)$

$d^n d^{n-1} \cdots d^{n-N+1} = 0$

- Objects: $\to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$
- Morphisms: Cochain maps.

Homotopy category of an additive subcategory $\mathcal{A}$: $K_N(\mathcal{A})$

- Objects: N-complexes in $\mathcal{A}$
- Morphisms: Homotopy equivalence classes

HOMOTOPY EQUIVALENCE

If $X, Y \in C_N(R)$ and $f, g : X \longrightarrow Y$ then

$$f \sim g \iff \exists\, s^n : X^n \longrightarrow Y^{n-N+1} \text{ such that}$$

$$f^n - g^n = \sum_{j=0}^{N-1} d_Y^{\{j\}} s^{i+N-j-1} d_X^{\{N-j-1\}}$$

Case $N = 3$

$$\begin{tikzcd}
X^{n-2} \arrow[r] & X^{n-1} \arrow[r] & X^{n} \arrow[r] \arrow[d] \arrow[dll, "s^{n}"'] & X^{n+1} \arrow[r] \arrow[dll, "s^{n+1}"'] & X^{n+2} \arrow[dll, "s^{n+2}"'] \\
Y^{n-2} \arrow[r] & Y^{n-1} \arrow[r] & Y^{n} \arrow[r] & Y^{n+1} \arrow[r] & Y^{n+2}
\end{tikzcd}$$

$$\boxed{\begin{aligned} f^n - g^n &= \\ &= d^{n-1} d^{n} s^{n} \\ &+ d^{n-1} s^{n+1} d^{n} \\ &+ s^{n+2} d^{n+1} d^{n} \end{aligned}}$$

2.1. COREFLECTIVE SUBCATEGORIES OF $K_N(\mathrm{Mod}\text{-}R)$

QUESTION: How can we construct coreflective subcategories of $K_N(\mathrm{Mod}\text{-}R)$?

Following Saorin - Stovicek

- We take a deconstructible class $\mathcal{A} \subseteq | \quad -R$
- Construct subcategories of $K(\mathrm{Mod}\text{-}R)$

a) $K(\mathcal{A})$ (Remember: N-complexes from $\mathcal{A}$)

b) $\underline{E(\mathcal{A})}$: N-acyclic complex from $\mathcal{A}$

N-ACYCLIC COMPLEXES

X is N-acyclic if all N-homology modules vanish.

Case $N=3$

Given an N-complex

$$X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^{n} \xrightarrow{d^{n}} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2}$$

$$d^n d^{n-1} d^{n-2} = 0 \implies \operatorname{Im} d^{n-2} d^{n-1} \leq \operatorname{Ker} d^n$$

$$d^{n+1} d^n d^{n-1} = 0 \implies \operatorname{Im} d^{n-1} \leq \operatorname{Ker} d^{n+1} d^n$$

X is N-acyclic if $\operatorname{Im} d^{n-2} d^{n-1} = \operatorname{Ker} d^n$ and $\operatorname{Im} d^{n-1} = \operatorname{Ker} d^{n+1} d^n$

2.1. COREFLECTIVE SUBCATEGORIES

Theorem

If $\mathcal{A} \subseteq \text{Mod-}R$ is deconstructible $\Longrightarrow$ $\mathbb{K}(\mathcal{A})$ and $\underline{E(\mathcal{A})}$ are coreflective subcategories of $\mathbb{K}(\text{Mod-}R)$

Proof

$\mathcal{A}$ deconstructible $\overset{①}{\Longrightarrow}$ $C_{\aleph}(\mathcal{A})$ and $E_{\aleph}(\mathcal{A})$ are deconstructible in $C_{\aleph}(R)$

$\overset{②}{\Longrightarrow}$ $C(\mathcal{A})$ and $E(\mathcal{A})$ are precovering in $C(R)$

$\overset{③}{\Longrightarrow}$ $\mathbb{K}(\mathcal{A})$ and $\underline{E(\mathcal{A})}$ are coreflective in $\mathbb{K}(\text{Mod-}R)$ $\#$

2.1. COREFLECTIVE SUBCATEGORIES

1. $C(\mathcal{A})$ and $E(\mathcal{A})$ are deconstructible.

If $X \in C(\mathcal{A})$, we have to find some filtration of X

$$(X_\alpha \mid \alpha < \kappa)$$

$$\begin{tikzcd}
X_0: & \cdots \arrow[r] & X_0^{n-1} \arrow[r] \arrow[d, hook] & X_0^{n} \arrow[r] \arrow[d, hook] & X_0^{n+1} \arrow[r] \arrow[d, hook] & \cdots \\
X_1: & \cdots \arrow[r] & X_1^{n-1} \arrow[r] \arrow[d] & X_1^{n} \arrow[r] \arrow[d] & X_1^{n+1} \arrow[r] \arrow[d] & \cdots \\
\frac{X_1}{X_0} & \cdots \arrow[r] & \frac{X_1^{n-1}}{X_0^{n-1}} \arrow[r] & \frac{X_1^{n}}{X_0^{n}} \arrow[r] & \frac{X_1^{n+1}}{X_0^{n+1}} \arrow[r] & \cdots
\end{tikzcd}$$

$\longrightarrow$ Belonging to some set of complexes.

Deconstruction

Hill's lemma

2.1. Coreflective subcategories

② $\mathcal{C}_{\mathcal{N}}(\mathcal{A})$ and $\mathcal{E}_{\mathcal{N}}(\mathcal{A})$ are precovering

Theorem (Enochs, Saorín–Šťovíček)

Every deconstructible class is precovering

Precovering class $\mathcal{X}$ in a category $\mathcal{C}$

For every $C \in \mathcal{C}$ exists a morphism $\varphi : X \to C$ with

a) $X \in \mathcal{X}$

b) Lifting property

$$\begin{tikzcd} X \arrow[r, "\varphi"] & C \\ & X' \in \mathcal{X} \arrow[ul, "f"] \arrow[u, "\psi"'] \arrow[ul, phantom, shift right=4, "\equiv"] \end{tikzcd}$$

$\varphi f = \psi$

2.1. COREFLECTIVE SUBCATEGORIES

3 $K_N(\mathcal{A})$ and $\underline{E_N(\mathcal{A})}$ are coreflective in $K_N(\text{Mod-}R)$

3.1 $K_N(\mathcal{A})$ and $\underline{E_N(\mathcal{A})}$ are precovering in $K_N(\text{Mod-}R)$

Theorem (Cortés-Izurdiaga-Crivei-Saorín, 2022)

If $\mathcal{A} \subseteq \mathcal{C}$ is a subcategory of an additive category with split idempotents, then $\mathcal{A}$ is coreflective $\Longleftrightarrow$

a) $\mathcal{A}$ is precovering.

b) $\mathcal{A}$ is closed under direct summands.

c) Every morphism in $\mathcal{A}$ has a pseudocokernel in $\mathcal{C}$ which belongs to $\mathcal{A}$.

2.2. $K_N(\mathrm{Proj})$ IS $\aleph_1$-COMPACTLY GENERATED

Theorem

$K_N(\mathrm{Proj})$ is $\aleph_1$-compactly generated.

Proof

The proof uses deconstruction and Hill lemma but with some particularities.

a) Proj is more than deconstructible: decomposable.

Kaplansky's Theorem

$$P \in \mathrm{Proj} \implies P = \bigoplus_{i=I} P_i \text{ with } P_i \text{ countably generated}$$

2.2. $K_{\aleph}(\mathrm{Proj})$ is $\aleph_1$-compactly generated

b) $C_{\aleph}(\mathrm{Proj})$ is more than deconstructible

If $X \in C_{\aleph}(\mathrm{Proj})$, the the filtration $(X_\alpha \mid \alpha < \kappa)$

$$
\begin{tikzcd}
X_0 : \arrow[d, hook] & \cdots \arrow[r] & X_0^{n-1} \arrow[r] \arrow[d, hook] & X_0^{n} \arrow[r] \arrow[d, hook] & X_0^{n+1} \arrow[r] \arrow[d, hook] & \cdots & \text{split mono.} \\
X_1 : \arrow[d] & \cdots \arrow[r] & X_1^{n-1} \arrow[r] \arrow[d] & X_1^{n} \arrow[r] \arrow[d] & X_1^{n+1} \arrow[r] \arrow[d] & \cdots & \text{split epim.} \\
\dfrac{X_1}{X_0} & \cdots \arrow[r] & \dfrac{X_1^{n-1}}{X_0^{n-1}} \arrow[r] & \dfrac{X_1^{n}}{X_0^{n}} \arrow[r] & \dfrac{X_1^{n+1}}{X_0^{n+1}} \arrow[r] & \cdots &
\end{tikzcd}
$$

$\longrightarrow$ countably generated projective

SPLITTING MORPHISMS

SPLIT MONOMORPHISM

$$
\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[l, bend left, dashed, "g"]
\end{tikzcd}
$$

$$g f(x) = x, \ \forall x \in M$$

$$g f = 1_M$$

SPLIT EPIMORPHISM

$$
\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[l, bend left, dashed, "s"]
\end{tikzcd}
$$

$$f s(x) = x, \ \forall x \in N$$

$$f s = 1_N$$

3. EXISTENCE OF GP-PRECOVERS

- We work in $\mathbf{K}(\mathrm{Proj})$ and consider $\mathcal{K} = \underline{\mathcal{X}} \subseteq \mathbf{K}(\mathrm{Proj})$ where $\mathcal{X}$ are the totally acyclic complexes.

Remember

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^{n} \xrightarrow{d^{n}} X^{n+1} \longrightarrow \cdots \quad \left\{ \begin{array}{l} \bullet\ X^{n} \text{ is projective} \\ \bullet\ \text{Acyclic: } \operatorname{Ker} d^{n} = \operatorname{Im} d^{n-1} \\ \bullet\ \text{Totally acyclic} \end{array} \right.$$

$$\begin{tikzcd} X^{n-1} \arrow[r, "d^{n-1}"] & X^{n} \arrow[r, "d^{n}"] \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "="] & X^{n+1} \arrow[dl, dashed, "g"] \\ & P & \end{tikzcd}$$

$gd^{n} = f$

$P \in \mathrm{Proj}$, $fd^{n} = 0$

3. EXISTENCE OF GP-PRECOVERS

Class of morphisms $\mathrm{Mor}_{\mathcal{K}}$

- Take a morphism $f : X \to Y$ in $\mathbb{K}(\mathrm{Proj})$
- By the axioms of triangulated categories

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

- Then $f \in \mathrm{Mor}_{\mathcal{K}} \iff Z \in \mathcal{K}$

Relevance of $\mathrm{Mor}_{\mathcal{K}}$

It allows to define the Verdier quotient

$$\mathbb{K}(\mathrm{Proj}) / \mathcal{K}$$

3. EXISTENCE OF GP-PRECOVERS

Theorem

If $\mathrm{Mor}_{\mathbb{K}}$ satisfies the generalized Baer lemma in $\mathbb{K}(\mathrm{Proj})$, then there exists GP-precovers in Mod-R.

$\mathrm{Mor}_{\mathbb{K}}$-injective objects

$X \in \mathbb{K}(\mathrm{Proj})$ is $\mathrm{Mor}_{\mathbb{K}}$-injective if

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\
X &
\end{tikzcd}
\qquad f \in \mathrm{Mor}_{\mathbb{K}}
$$

$$hf = g$$

3. EXISTENCE OF GP-PRECOVERS

Generalized Baer Lemma

$\mathrm{Mor}_{\mathcal{K}}$ satisfies the generalized Baer lemma if there is a set $\mathcal{N} \subseteq \mathrm{Mor}_{\mathcal{K}}$ (not a class!) such that

$$X \text{ is } \mathrm{Mor}_{\mathcal{K}}\text{-injective} \iff X \text{ is } \mathcal{N}\text{-injective}$$

Example

- $M \in \mathrm{Mod}\text{-}R$ is injective $\iff$ M is injective w.r.t. any monomorphism
- **Classical Baer lemma**

M is injective $\iff M$ is $\mathcal{N}$-injective

$\mathcal{N}$ = Monomorphisms $f : I \to R$

3. EXISTENCE OF GP-PRECOVERS

Theorem (CI-Guil-Kaleboğaz-Srivastava, 2020)

If $(\mathcal{C};\mathcal{E})$ is an exact category satisfying certain conditions such that the class of inflations satisfy the Generalized Baer lemma, then

$(\mathcal{C};\mathcal{E})$ has enough injectives.

3. EXISTENCE OF GP-PRECOVERS

Theorem

If $\mathrm{Mor}_{\mathbb{K}}$ satisfies the generalized Baer lemma in $\mathbb{K}(\mathrm{Proj})$, then there exists GP-precovers in Mod-R.

Proof

a) The Verdier quotient $\mathbb{K}(\mathrm{Proj})/\mathbb{K}$ has small hom-sets.

b) $\mathbb{K}$ is coreflective in $\mathbb{K}(\mathrm{Proj})$

Using results from triangulated categories

c) There exist GP-precovers

By Jorgensen result $\#$

3. EXISTENCE OF GP-PRECOVERS

a) The Verdier quotient $\mathbb{K}(\mathrm{Proj})/\mathcal{K}$ has small hom-sets.

- The category $\mathbb{K}(\mathrm{Proj})/\mathcal{K}$
    - Objects: $\mathbb{K}(\mathrm{Proj})$
    - Morphisms between X and Y
      Equivalent classes of triples (Z, f, g)

It need not be a set!

$$\mathrm{Mor}_{\mathcal{K}} \ni \begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & Z'' \arrow[l] \arrow[r] \arrow[u, dashed] \arrow[d, dashed] & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

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THANK YOU VERY MUCH!