A Tale of Two Sequences

Vitor Gulisz

Northeastern University

April 12, 2024

Some background

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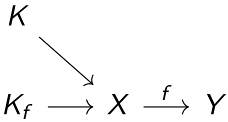

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$$\begin{array}{ccccc} K & & & & \\ {\scriptstyle \exists!}\downarrow & \searrow & & & \\ K_f & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

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$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & L_f \\ & & & \searrow & \\ & & & & L \end{array}$$

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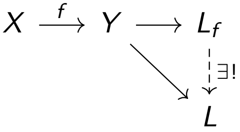

Snake Lemma. If

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence given by the dashed arrows below.

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$$\begin{array}{ccccc}
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\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\
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\end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence given by the dashed arrows below.

$$\begin{array}{ccccccccc}
 & & K_r & & K_g & & K_s & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
K_f & \rightarrowtail & A & \xrightarrow{f} & B & \twoheadrightarrow & C & & \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} & & \\
 & & D & \rightarrowtail & E & \xrightarrow[h]{} & F & \twoheadrightarrow & L_h \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L_r & & L_g & & L_s & &
\end{array}$$

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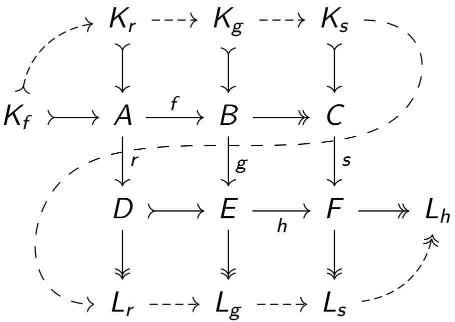

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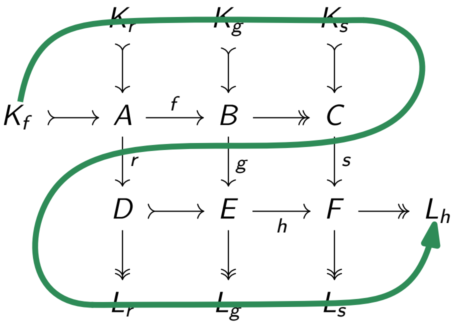
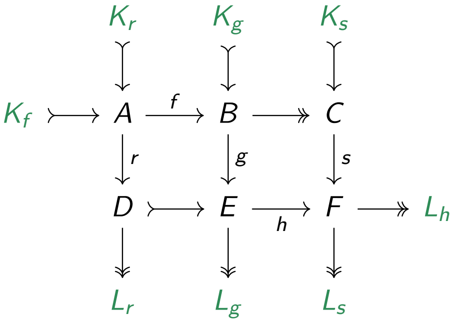


First proof

By diagram chase

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle s} \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$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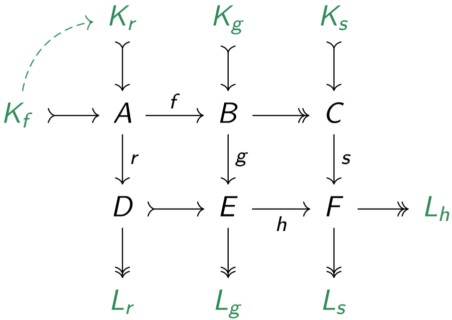
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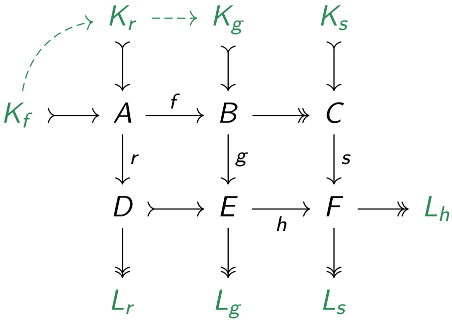
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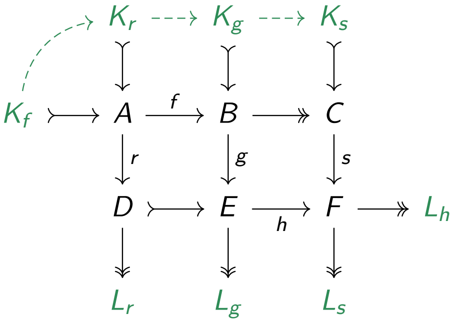
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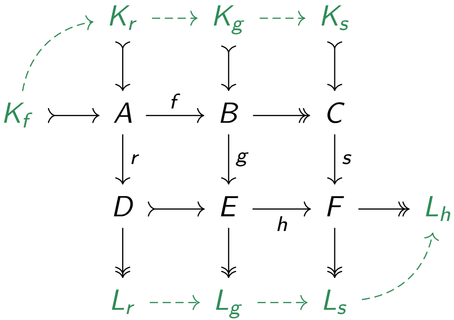

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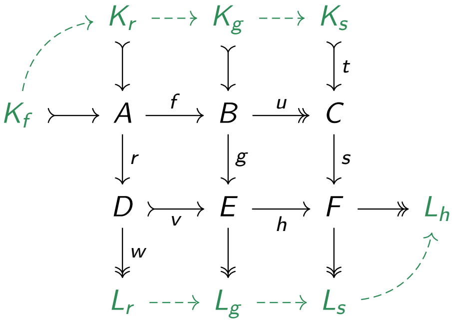

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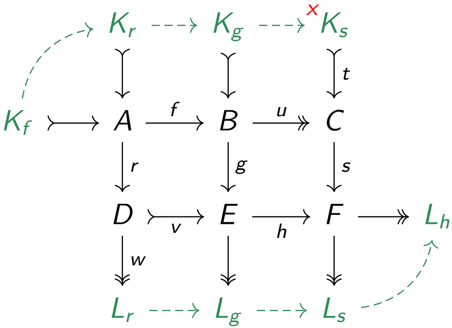
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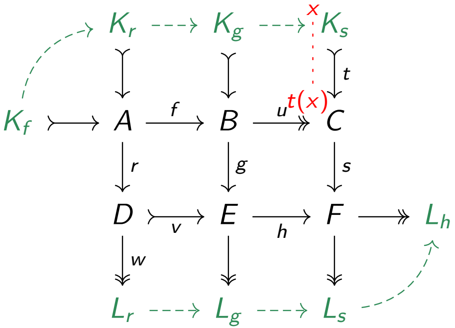

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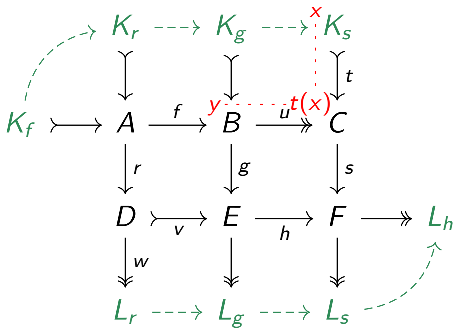

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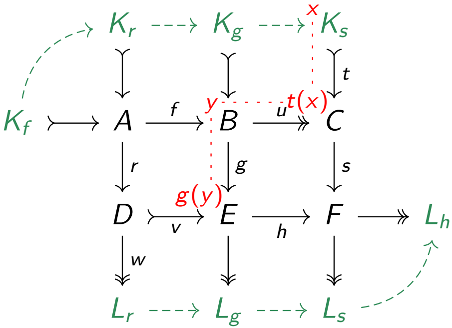

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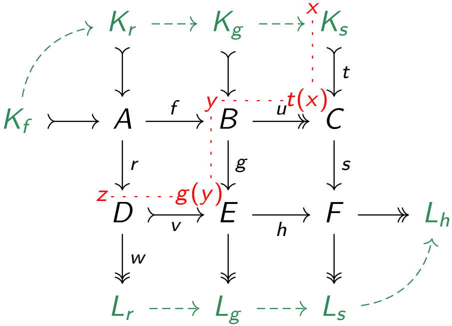
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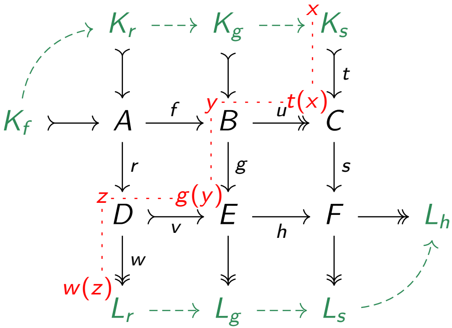

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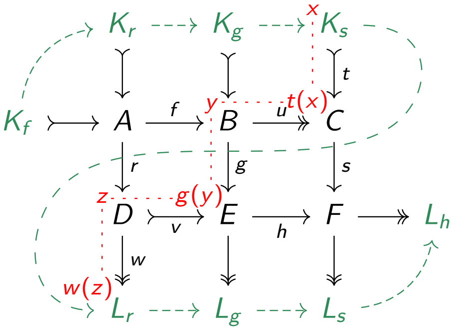
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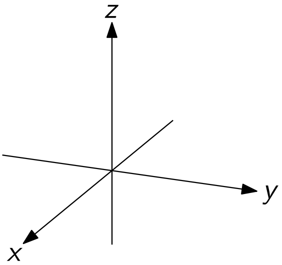
An example for vector spaces over $\mathbb{R}$

$$\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} & \mathbb{R}^3 & & \\
 & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}} & & \\
 & & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}]{} & \mathbb{R}
\end{array}$$

$$\mathbb{R} \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} \mathbb{R}^3$$

$$\downarrow \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$

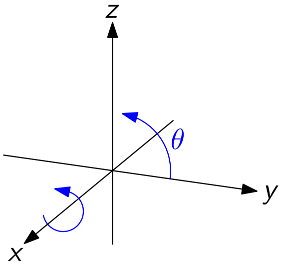
$$\mathbb{R}^3 \xrightarrow[(1\ 0\ 0)]{} \mathbb{R}$$



$$\mathbb{R} \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} \mathbb{R}^3$$

$$\downarrow \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$

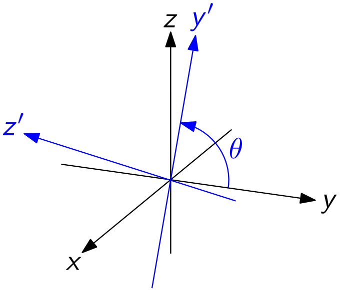
$$\mathbb{R}^3 \xrightarrow[(1\ 0\ 0)]{} \mathbb{R}$$



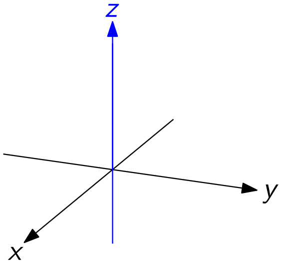
$$\mathbb{R} \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} \mathbb{R}^3$$

$$\downarrow \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$

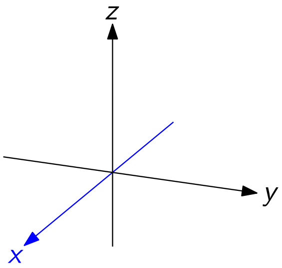
$$\mathbb{R}^3 \xrightarrow[(\,1\ 0\ 0\,)]{} \mathbb{R}$$



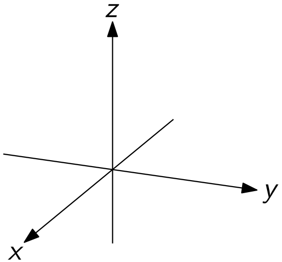
$$\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & & \\
 & & \Big\downarrow \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{pmatrix} & & \\
 & & \mathbb{R}^3 & \xrightarrow[(\,1\;0\;0\,)]{} & \mathbb{R}
\end{array}$$



$$
\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} & \mathbb{R}^3 & & \\
 & & \Big\downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} & & \\
 & & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}]{} & \mathbb{R}
\end{array}
$$



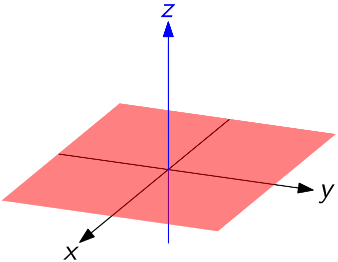
$$
\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} & \mathbb{R}^2 \\
 & & \downarrow \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix} & & \\
\mathbb{R}^2 & \xrightarrow[\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}]{} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix}1&0&0\end{pmatrix}]{} & \mathbb{R}
\end{array}
$$



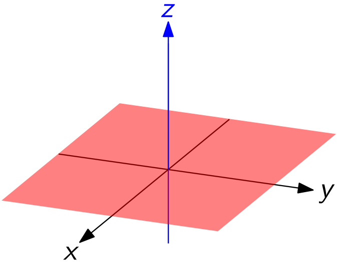
$$\mathbb{R} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \mathbb{R}^2$$

$$\downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbb{R}^2 \xrightarrow[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}]{} \mathbb{R}^3 \xrightarrow[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}]{} \mathbb{R}$$



$$\begin{array}{ccccc}
\mathbb{R}_z & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & \overset{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}}{\twoheadrightarrow} & \mathbb{R}^2_{xy} \\
 & & \Big\downarrow \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix} & & \\
\mathbb{R}^2 & \underset{\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}}{\rightarrowtail} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix}1&0&0\end{pmatrix}]{} & \mathbb{R}
\end{array}$$



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\begin{array}{ccccc}
\mathbb{R}_z & \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} & \mathbb{R}^3 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} & \mathbb{R}^2_{xy} \\
 & & \Big\downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} & & \\
\mathbb{R}^2 & \xrightarrow[\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}]{} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}]{} & \mathbb{R}
\end{array}
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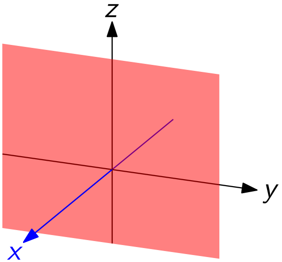

$$
\begin{array}{ccccc}
\mathbb{R}_z & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} & \mathbb{R}^2_{xy} \\
 & & \Big\downarrow \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix} & & \\
\mathbb{R}^2_{yz} & \xrightarrow[\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}]{} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix}1&0&0\end{pmatrix}]{} & \mathbb{R}_x
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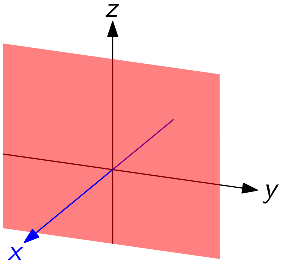

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\mathbb{R}_z & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & \overset{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}}{\twoheadrightarrow} & \mathbb{R}^2_{xy} \\
 & & \Big\downarrow \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix} & & \\
\mathbb{R}^2_{yz} & \underset{\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}}{\rightarrowtail} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix}1&0&0\end{pmatrix}]{} & \mathbb{R}_x
\end{array}$$

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\mathbb{R}_z & \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} & \mathbb{R}^3 & \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} & \mathbb{R}^2_{xy} \\
\left\downarrow{\scriptstyle \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}}\right. & & \left\downarrow{\scriptstyle \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}}\right. & & \left\downarrow{\scriptstyle \begin{pmatrix}1&0\end{pmatrix}}\right. \\
\mathbb{R}^2_{yz} & \xrightarrow[\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}]{} & \mathbb{R}^3 & \xrightarrow[\begin{pmatrix}1&0&0\end{pmatrix}]{} & \mathbb{R}_x
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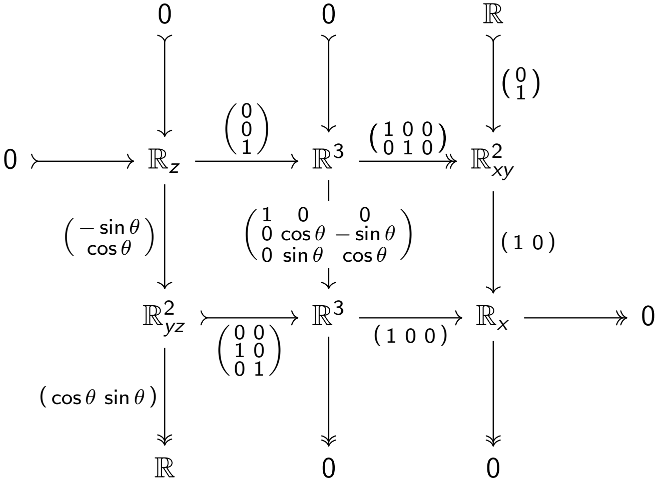
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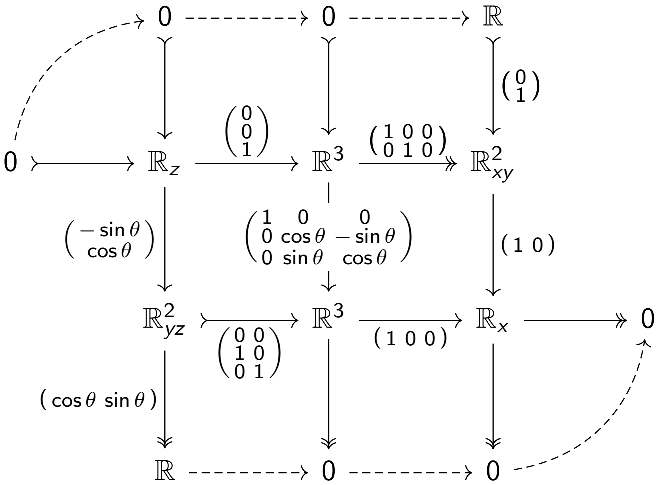
0 0 $\mathbb{R}$
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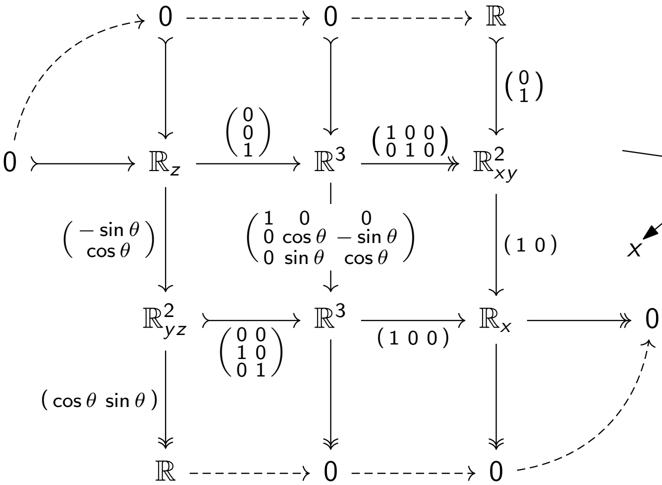
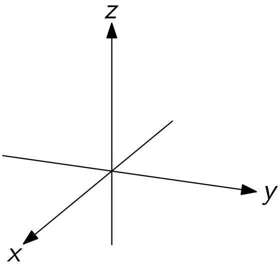
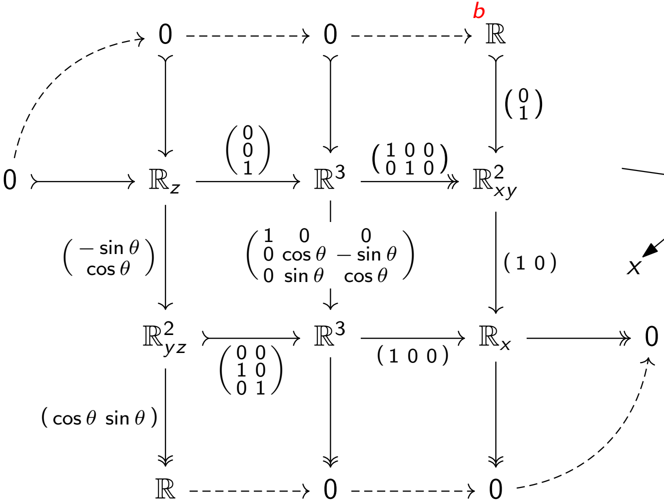
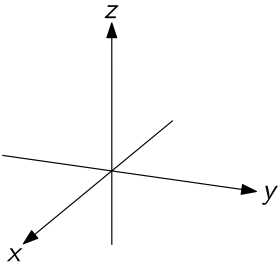
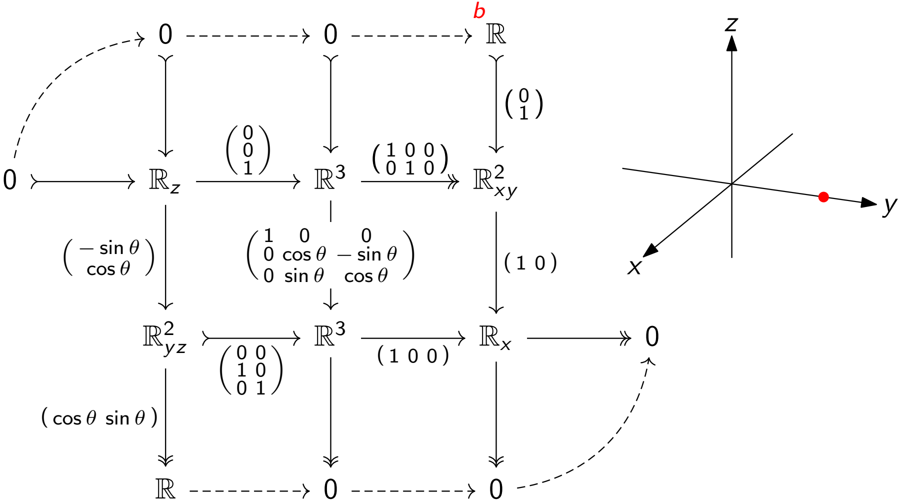
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\mathbb{R}^2_{yz}$ $\mathbb{R}^3$ $\mathbb{R}_x$ 0
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$
$\mathbb{R}$ 0 0

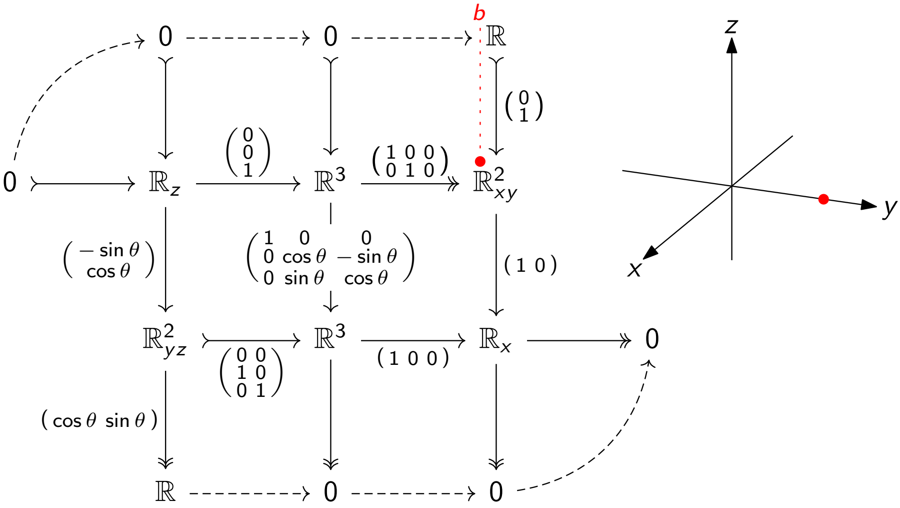

0
0
ℝ
0
$\mathbb{R}_z$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$
$\mathbb{R}^2_{xy}$
$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0\end{pmatrix}$
$\mathbb{R}^2_{yz}$
$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix}1&0&0\end{pmatrix}$
$\mathbb{R}_x$
0
$\begin{pmatrix}\cos\theta&\sin\theta\end{pmatrix}$
ℝ
0
0


z
y
x

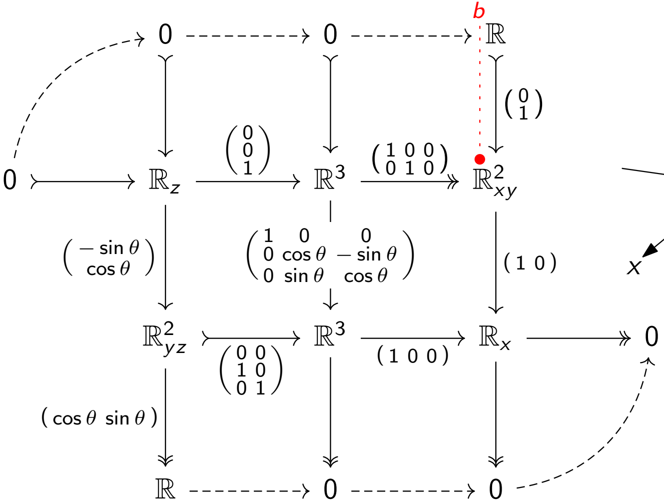

b
0 0 $\mathbb{R}$
0 $\mathbb{R}_z$ $\mathbb{R}^3$ $\mathbb{R}^2_{xy}$
$\mathbb{R}^2_{yz}$ $\mathbb{R}^3$ $\mathbb{R}_x$ 0
$\mathbb{R}$ 0 0
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$

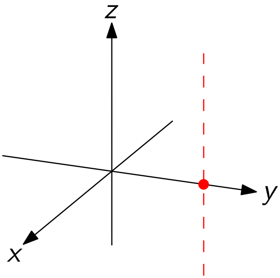

z
y
x

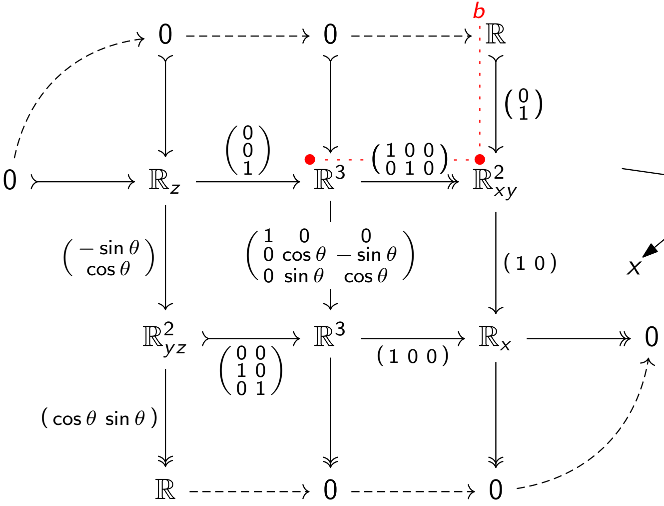

b
0
0
ℝ
$\begin{pmatrix}0\\1\end{pmatrix}$
0
$\mathbb{R}_z$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$
$\mathbb{R}^2_{xy}$
$\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0\end{pmatrix}$
$\mathbb{R}^2_{yz}$
$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix}1&0&0\end{pmatrix}$
$\mathbb{R}_x$
0
$\begin{pmatrix}\cos\theta&\sin\theta\end{pmatrix}$
ℝ
0
0
z
y
x

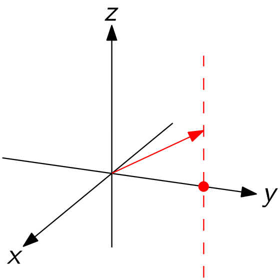


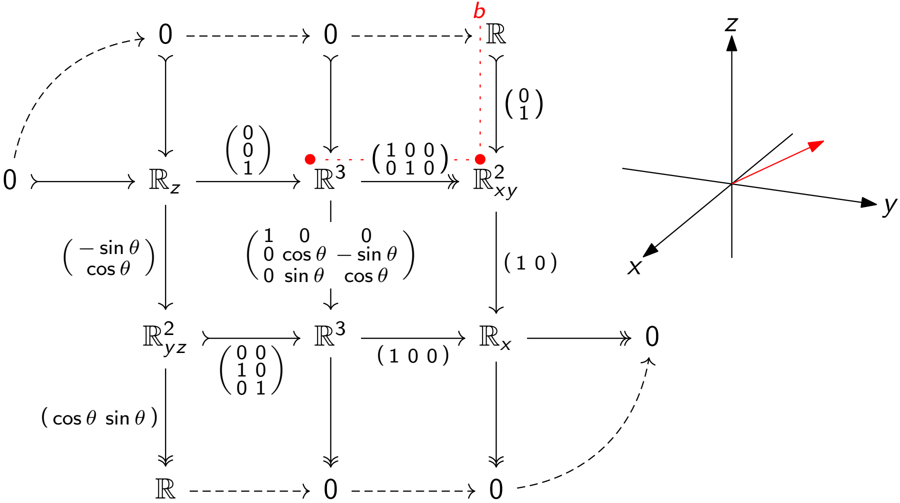
b
0 → 0 → ℝ
0 → $\mathbb{R}_z$ → $\mathbb{R}^3$ → $\mathbb{R}^2_{xy}$
$\mathbb{R}^2_{yz}$ → $\mathbb{R}^3$ → $\mathbb{R}_x$ → 0
ℝ → 0 → 0
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$
z
y
x


b
0
0
ℝ
0
ℝ_z
ℝ^3
ℝ^2_xy
ℝ^2_yz
ℝ^3
ℝ_x
0
ℝ
0
0
(0 0 1)
(1 0 0 / 0 1 0)
(0 1)
(−sin θ / cos θ)
(1 0 0 / 0 cos θ −sin θ / 0 sin θ cos θ)
(1 0)
(0 0 / 1 0 / 0 1)
(1 0 0)
(cos θ sin θ)

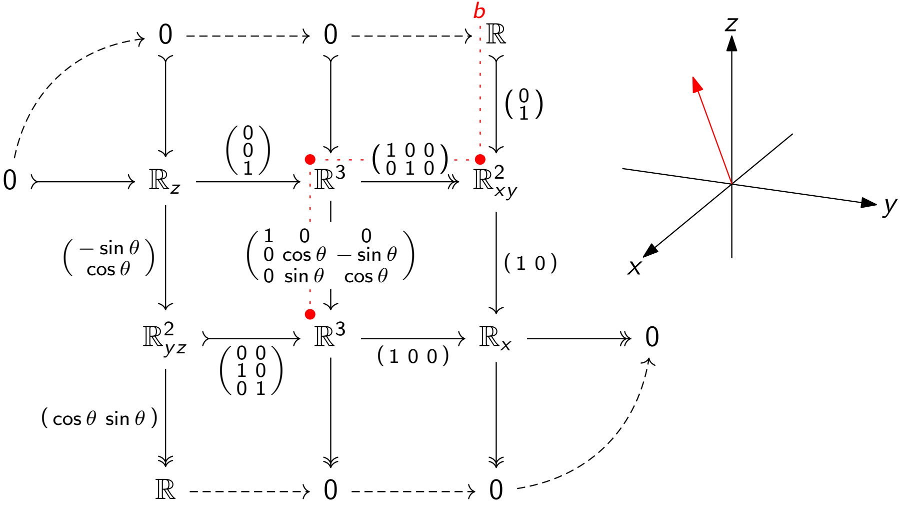

z
y
x

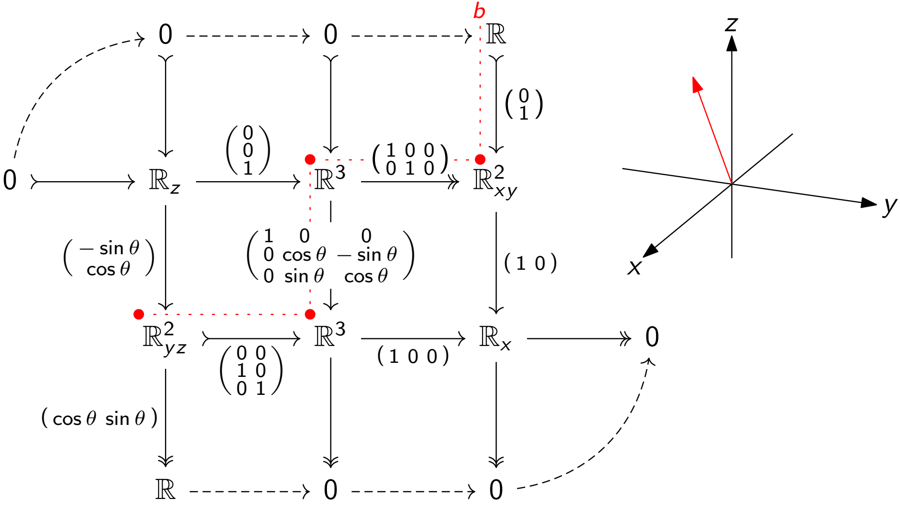

b
0 0 ℝ
0 ℝ_z ℝ^3 ℝ^2_xy
(0 0 1)
(1 0 0 / 0 1 0)
(0 1)
(−sin θ / cos θ)
(1 0 0 / 0 cos θ −sin θ / 0 sin θ cos θ)
(1 0)
ℝ^2_yz ℝ^3 ℝ_x 0
(0 0 / 1 0 / 0 1)
(1 0 0)
(cos θ sin θ)
ℝ 0 0

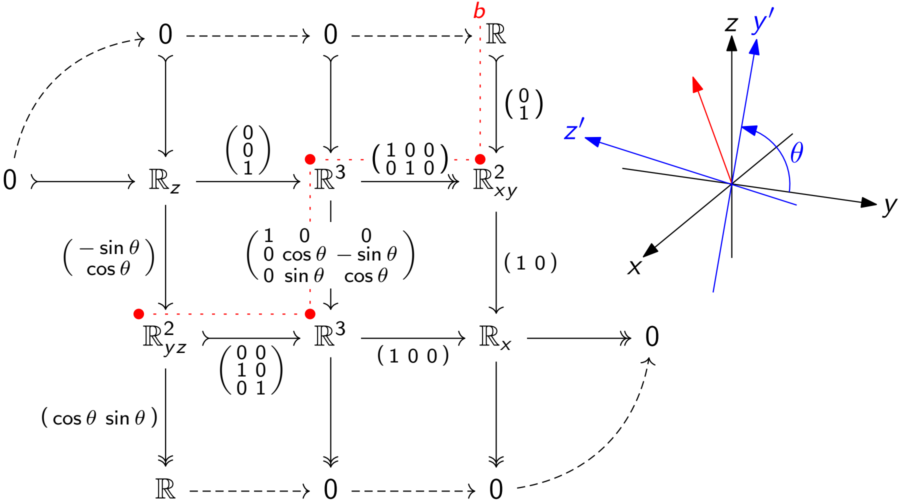

z
y
x



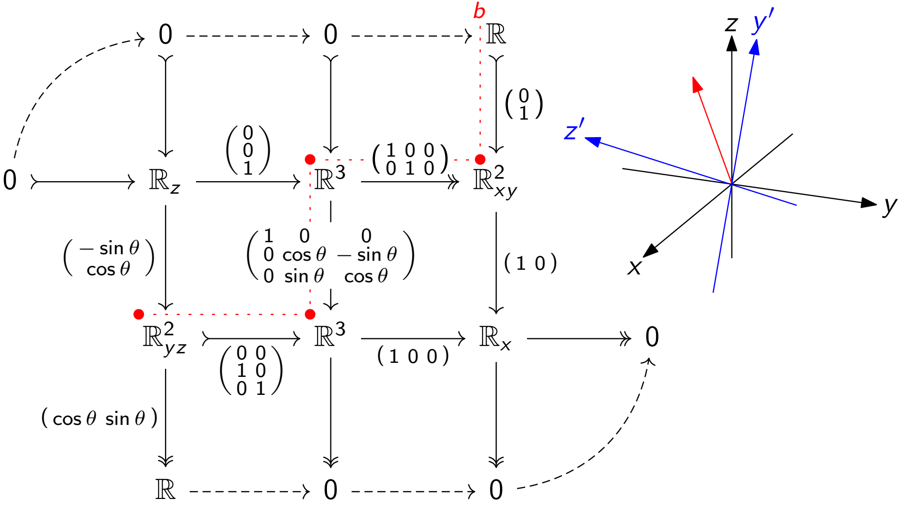
b
0
0
$\mathbb{R}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
0
$\mathbb{R}_z$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\mathbb{R}^2_{xy}$
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\mathbb{R}^2_{yz}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\mathbb{R}_x$
0
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$
$\mathbb{R}$
0
0
z
y
x

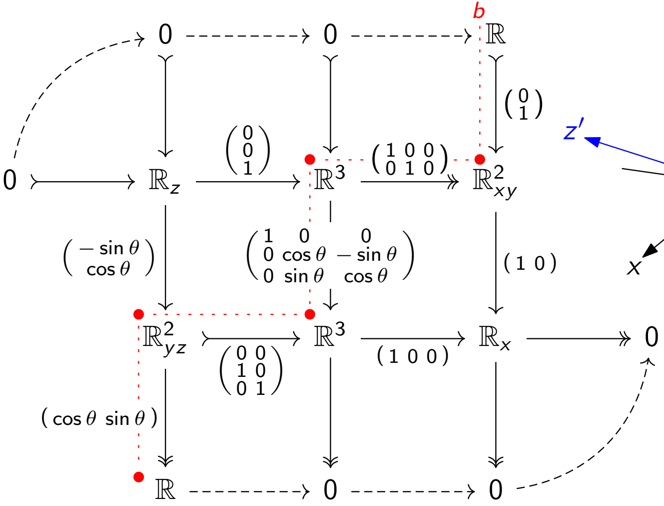

b
0
0
ℝ
0
$\mathbb{R}_z$
$\mathbb{R}^3$
$\mathbb{R}^2_{xy}$
$\mathbb{R}^2_{yz}$
$\mathbb{R}^3$
$\mathbb{R}_x$
0
ℝ
0
0
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$
z
y
x

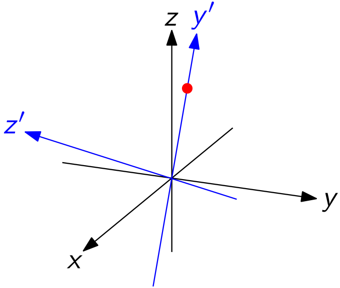

b
0
0
ℝ
$\begin{pmatrix}0\\1\end{pmatrix}$
0
$\mathbb{R}_z$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\mathbb{R}^3$
$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$
$\mathbb{R}^2_{xy}$
$\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0\end{pmatrix}$
$\mathbb{R}^2_{yz}$
$\mathbb{R}^3$
$\mathbb{R}_x$
0
$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$
$\begin{pmatrix}1&0&0\end{pmatrix}$
$\begin{pmatrix}\cos\theta&\sin\theta\end{pmatrix}$
ℝ
0
0
z
y
x

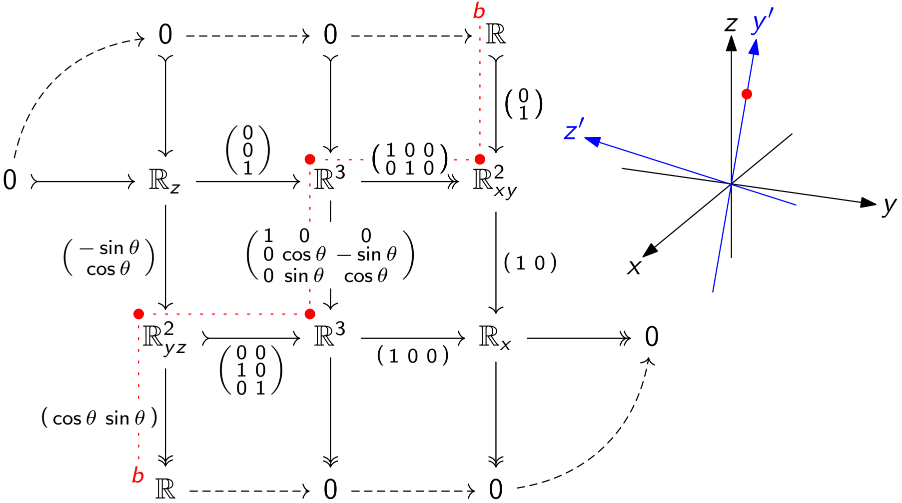


b
0 → 0 → ℝ
0 → $\mathbb{R}_z$ → $\mathbb{R}^3$ → $\mathbb{R}^2_{xy}$
$\mathbb{R}^2_{yz}$ → $\mathbb{R}^3$ → $\mathbb{R}_x$ → 0
ℝ → 0 → 0
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$
z
y′
z′
θ
y
x


b
0
0
ℝ
0
ℝ_z
ℝ^3
ℝ^2_xy
ℝ^2_yz
ℝ^3
ℝ_x
0
ℝ
0
0
(0 0 1)
(1 0 0; 0 1 0)
(0 1)
(−sin θ; cos θ)
(1 0 0; 0 cos θ −sin θ; 0 sin θ cos θ)
(1 0)
(0 0; 1 0; 0 1)
(1 0 0)
(cos θ sin θ)
z
y′
z′
y
x


b
0
ℝ
$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$
$\mathbb{R}_z$
$\mathbb{R}^3$
$\mathbb{R}^2_{xy}$
$\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$
$\begin{pmatrix}1&0\end{pmatrix}$
$\mathbb{R}^2_{yz}$
$\mathbb{R}_x$
$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$
$\begin{pmatrix}1&0&0\end{pmatrix}$
$\begin{pmatrix}\cos\theta&\sin\theta\end{pmatrix}$
$\mathbb{R}$


z
y'
z'
y
x

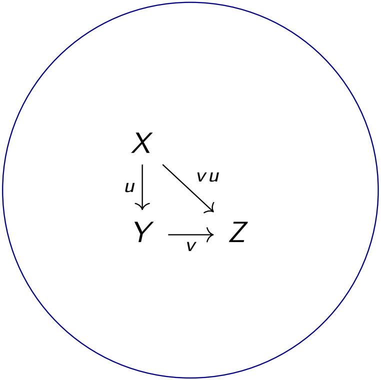

b
0
0
ℝ
(0 1)
0
ℝ_z
(0 0 1)
ℝ^3
(1 0 0; 0 1 0)
ℝ^2_xy
(−sin θ; cos θ)
(1 0 0; 0 cos θ −sin θ; 0 sin θ cos θ)
(1 0)
ℝ^2_yz
ℝ^3
ℝ_x
0
(0 0; 1 0; 0 1)
(1 0 0)
(cos θ sin θ)
b
ℝ
0
0
z
y′
z′
y
x

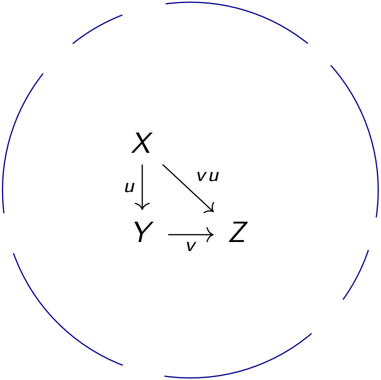
The circular sequence

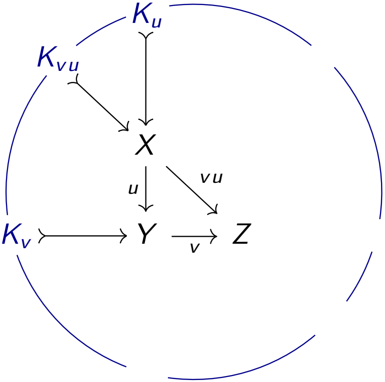
$$X \xrightarrow[u]{} Y \xrightarrow[v]{} Z$$

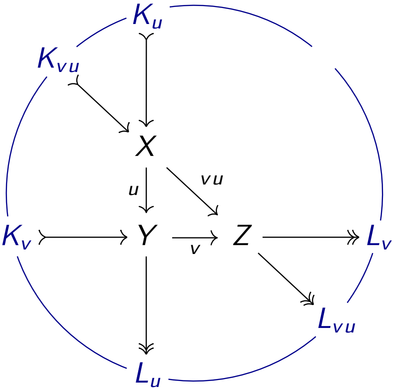
$$\begin{array}{ccc} X & & \\ {\scriptstyle u}\downarrow & & \\ Y & \xrightarrow[v]{} & Z \end{array}$$

$$
\begin{array}{ccc}
X & & \\
{\scriptstyle u}\downarrow & \searrow^{vu} & \\
Y & \xrightarrow[v]{} & Z
\end{array}
$$

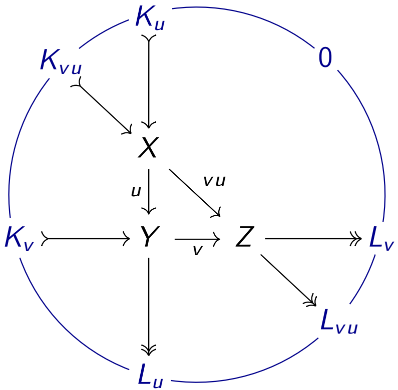

X
u
vu
Y
v
Z

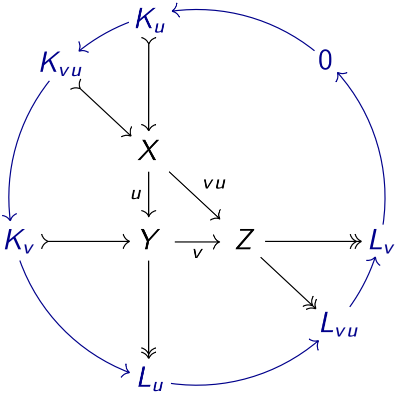

X
u
vu
Y
v
Z


K_u
K_{vu}
X
u
vu
K_v
Y
v
Z



K_u
K_{vu}
X
u
vu
K_v
Y
v
Z
L_v
L_{vu}
L_u


K_u
K_{vu}
0
X
u
vu
K_v
Y
v
Z
L_v
L_{vu}
L_u


K_u
K_{vu}
0
X
u
vu
K_v
Y
v
Z
L_v
L_{vu}
L_u

The circular sequence (aka kernel-cokernel sequence). For every pair of composable morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \ ,$$

the induced sequence

$$0 \longrightarrow K_u \longrightarrow K_{vu} \longrightarrow K_v \longrightarrow L_u \longrightarrow L_{vu} \longrightarrow L_v \longrightarrow 0$$

is exact.

The circular sequence (aka kernel-cokernel sequence). For every pair of composable morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \ ,$$

the induced sequence

$$K_u \rightarrowtail K_{vu} \longrightarrow K_v \longrightarrow L_u \longrightarrow L_{vu} \twoheadrightarrow L_v$$

is exact.

- C. H. Dowker. Composite morphisms in abelian categories. *Quart. J. Math. Oxford Ser. (2)*, 17:98–105, 1966.
- Johann B. Leicht. Über die elementaren Lemmata der homologischen Algebra in quasi-exacten Kategorien. *Monatsh. Math.*, 68:240–254, 1964.

- C. H. Dowker. Composite morphisms in abelian categories. *Quart. J. Math. Oxford Ser. (2)*, 17:98–105, 1966.
- Johann B. Leicht. Über die elementaren Lemmata der homologischen Algebra in quasi-exacten Kategorien. *Monatsh. Math.*, 68:240–254, 1964.

5 Thanks! It surprises me that there are *two* fundamental diagram chase lemmas concerning the kernel and cokernel of maps, but that one of them is substantially more famous than the other... I had never encountered the kernel-cokernel exact sequence before, but have been exposed to the snake lemma countless times. – Terry Tao Dec 23, 2023 at 22:02

mathoverflow.net/questions/460926

Proposition. Any morphism

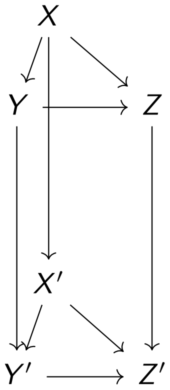
$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' \end{array}$$

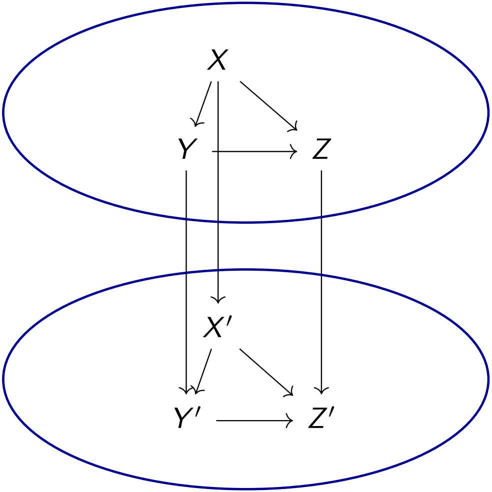
from vu to $v'u'$ induces a morphism for the corresponding circular sequences

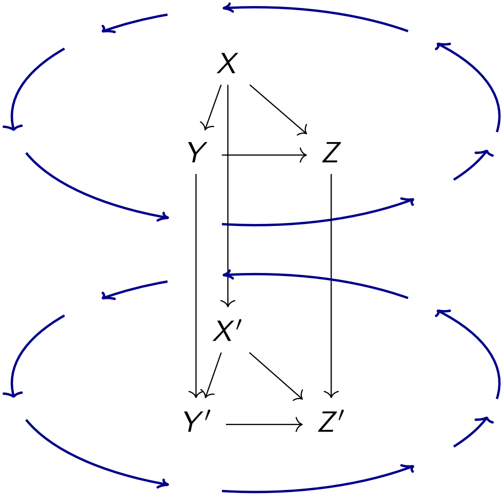
$$\begin{array}{ccccccccccc} K_u & \rightarrowtail & K_{vu} & \longrightarrow & K_v & \longrightarrow & L_u & \longrightarrow & L_{vu} & \twoheadrightarrow & L_v \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{u'} & \rightarrowtail & K_{v'u'} & \longrightarrow & K_{v'} & \longrightarrow & L_{u'} & \longrightarrow & L_{v'u'} & \twoheadrightarrow & L_{v'} \end{array} .$$

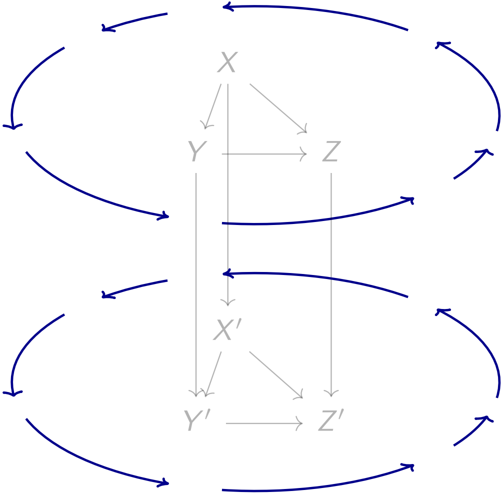
$$
\begin{array}{ccc}
 & X & \\
\swarrow & & \searrow \\
Y & \longrightarrow & Z
\end{array}
$$

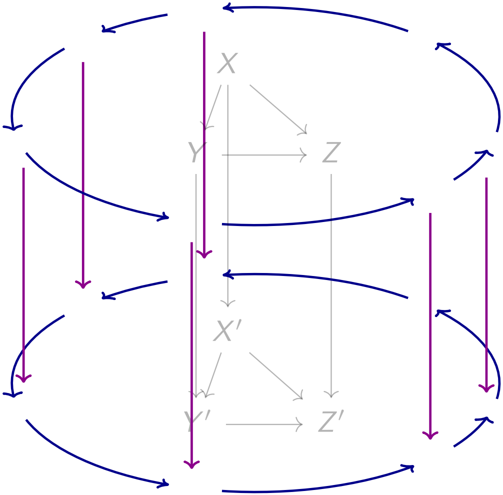
$$
\begin{array}{ccc}
 & X' & \\
\swarrow & & \searrow \\
Y' & \longrightarrow & Z'
\end{array}
$$


X
Y
Z
X'
Y'
Z'


X
Y
Z
X′
Y′
Z′


X
Y
Z
X′
Y′
Z′


X
Y
Z
X′
Y′
Z′

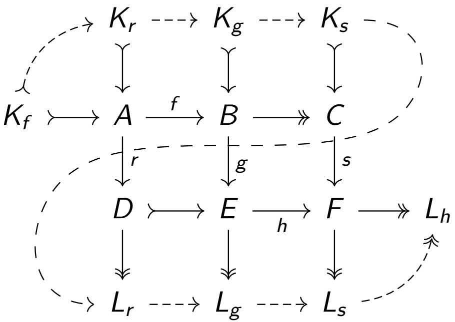

X
Y
Z
X′
Y′
Z′

$$\begin{array}{ccccccccccc}
K_u & \rightarrowtail & K_{vu} & \longrightarrow & K_v & \longrightarrow & L_u & \longrightarrow & L_{vu} & \twoheadrightarrow & L_v \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{u'} & \rightarrowtail & K_{v'u'} & \longrightarrow & K_{v'} & \longrightarrow & L_{u'} & \longrightarrow & L_{v'u'} & \twoheadrightarrow & L_{v'}
\end{array}$$

Snake Lemma. If

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$

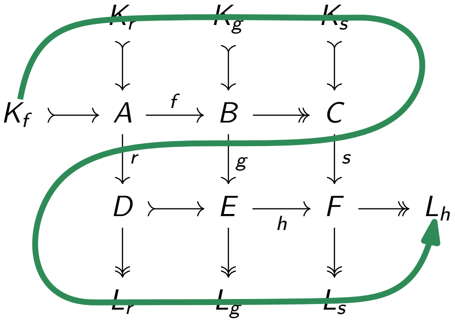
is a commutative diagram with exact rows, then there is an exact sequence given by the dashed arrows below.



Snake Lemma. If

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ \downarrow r & & \downarrow g & & \downarrow s \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence given by the dashed arrows below.



Second proof

Element-free, by using the circular sequence

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \twoheadrightarrow & C \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\
D & \rightarrowtail & E & \xrightarrow[h]{} & F
\end{array}
$$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle s} \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \twoheadrightarrow & C \\ & & \Big\downarrow{\scriptstyle g} & & \\ D & \rightarrowtail & E & \xrightarrow[h]{} & F \end{array}$$

$$hgf = 0$$

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \twoheadrightarrow & L_f \\
 & & \big\downarrow{\scriptstyle g} & & \\
K_h & \rightarrowtail & E & \xrightarrow[h]{} & F
\end{array}
$$

$$hgf = 0$$

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & & \\
 & & \Big\downarrow{\scriptstyle g} & & \\
 & & E & \xrightarrow[h]{} & F
\end{array}
$$

$$hgf = 0$$

Possible compositions for

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ & & \Big\downarrow g & & \\ & & E & \xrightarrow[h]{} & F \end{array}$$

Possible compositions for $\begin{array}{ccc} A & \xrightarrow{f} & B \\ & & \big\downarrow g \\ & & E & \xrightarrow[h]{} & F \end{array}$

$gf \qquad A \xrightarrow{f} B \xrightarrow{g} E$

Possible compositions for
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ & & \big\downarrow{\scriptstyle g} & & \\ & & E & \xrightarrow[h]{} & F \end{array}$$

$$gf \qquad A \xrightarrow{f} B \xrightarrow{g} E$$

$$(hg)f \qquad A \xrightarrow{f} B \xrightarrow{hg} F$$

Possible compositions for
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & & \big\downarrow g \\ & & E & \xrightarrow[h]{} & F \end{array}$$

$$gf \qquad A \xrightarrow{f} B \xrightarrow{g} E$$

$$(hg)f \qquad A \xrightarrow{f} B \xrightarrow{hg} F$$

$$h(gf) \qquad A \xrightarrow{gf} E \xrightarrow{h} F$$

Possible compositions for

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & & \downarrow g \\ & & E & \xrightarrow[h]{} & F \end{array}$$

$$gf \qquad A \xrightarrow{f} B \xrightarrow{g} E$$

$$(hg)f \qquad A \xrightarrow{f} B \xrightarrow{hg} F$$

$$h(gf) \qquad A \xrightarrow{gf} E \xrightarrow{h} F$$

$$hg \qquad B \xrightarrow{g} E \xrightarrow{h} F$$

Possible compositions for

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ & & \Big\downarrow{\scriptstyle g} & & \\ & & E & \xrightarrow[h]{} & F \end{array}$$

$$\begin{array}{lccccc}
gf & A & \xrightarrow{f} & B & \xrightarrow{g} & E \\
 & \Big\| & & \Big\| & & \Big\downarrow{\scriptstyle h} \\
(hg)f & A & \xrightarrow{f} & B & \xrightarrow{hg} & F \\
 & \Big\| & & \Big\downarrow{\scriptstyle g} & & \Big\| \\
h(gf) & A & \xrightarrow{gf} & E & \xrightarrow{h} & F \\
 & {\scriptstyle f}\Big\downarrow & & \Big\| & & \Big\| \\
hg & B & \xrightarrow{g} & E & \xrightarrow{h} & F
\end{array}$$

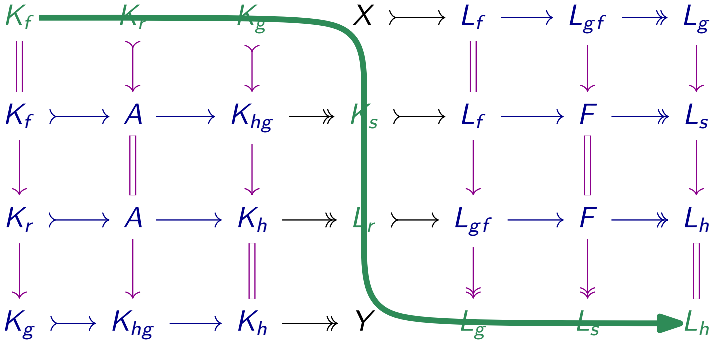
$$
\begin{array}{ccccccccccc}
K_f & \rightarrowtail & K_r & \longrightarrow & K_g & \longrightarrow & L_f & \longrightarrow & L_{gf} & \twoheadrightarrow & L_g \\
\| & & \downarrowtail & & \downarrowtail & & \| & & \downarrow & & \downarrow \\
K_f & \rightarrowtail & A & \longrightarrow & K_{hg} & \longrightarrow & L_f & \longrightarrow & F & \twoheadrightarrow & L_s \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| & & \downarrow \\
K_r & \rightarrowtail & A & \longrightarrow & K_h & \longrightarrow & L_{gf} & \longrightarrow & F & \twoheadrightarrow & L_h \\
\downarrow & & \downarrow & & \| & & \twoheaddownarrow & & \twoheaddownarrow & & \| \\
K_g & \rightarrowtail & K_{hg} & \longrightarrow & K_h & \longrightarrow & L_g & \longrightarrow & L_s & \twoheadrightarrow & L_h
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
K_f & \rightarrowtail & K_r & \longrightarrow & K_g & \twoheadrightarrow & X & \rightarrowtail & L_f & \longrightarrow & L_{gf} & \twoheadrightarrow & L_g \\
\parallel & & \downarrow & & \downarrow & & & & \parallel & & \downarrow & & \downarrow \\
K_f & \rightarrowtail & A & \longrightarrow & K_{hg} & \twoheadrightarrow & K_s & \rightarrowtail & L_f & \longrightarrow & F & \twoheadrightarrow & L_s \\
\downarrow & & \parallel & & \downarrow & & & & \downarrow & & \parallel & & \downarrow \\
K_r & \rightarrowtail & A & \longrightarrow & K_h & \twoheadrightarrow & L_r & \rightarrowtail & L_{gf} & \longrightarrow & F & \twoheadrightarrow & L_h \\
\downarrow & & \downarrow & & \parallel & & & & \downarrow & & \downarrow & & \parallel \\
K_g & \rightarrowtail & K_{hg} & \longrightarrow & K_h & \twoheadrightarrow & Y & \rightarrowtail & L_g & \longrightarrow & L_s & \twoheadrightarrow & L_h
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
K_f & \rightarrowtail & K_r & \longrightarrow & K_g & \twoheadrightarrow & X & \rightarrowtail & L_f & \longrightarrow & L_{gf} & \twoheadrightarrow & L_g \\
\parallel & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow \\
K_f & \rightarrowtail & A & \longrightarrow & K_{hg} & \twoheadrightarrow & K_s & \rightarrowtail & L_f & \longrightarrow & F & \twoheadrightarrow & L_s \\
\downarrow & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
K_r & \rightarrowtail & A & \longrightarrow & K_h & \twoheadrightarrow & L_r & \rightarrowtail & L_{gf} & \longrightarrow & F & \twoheadrightarrow & L_h \\
\downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \parallel \\
K_g & \rightarrowtail & K_{hg} & \longrightarrow & K_h & \twoheadrightarrow & Y & \rightarrowtail & L_g & \longrightarrow & L_s & \twoheadrightarrow & L_h
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
K_f & \rightarrowtail & K_r & \longrightarrow & K_g & \twoheadrightarrow & X & \rightarrowtail & L_f & \longrightarrow & L_{gf} & \twoheadrightarrow & L_g \\
\Vert & & \downarrow & & \downarrow & & \downarrow & & \Vert & & \downarrow & & \downarrow \\
K_f & \rightarrowtail & A & \longrightarrow & K_{hg} & \twoheadrightarrow & K_s & \rightarrowtail & L_f & \longrightarrow & F & \twoheadrightarrow & L_s \\
\downarrow & & \Vert & & \downarrow & & \downarrow & & \downarrow & & \Vert & & \downarrow \\
K_r & \rightarrowtail & A & \longrightarrow & K_h & \twoheadrightarrow & L_r & \rightarrowtail & L_{gf} & \longrightarrow & F & \twoheadrightarrow & L_h \\
\downarrow & & \downarrow & & \Vert & & \downarrow & & \downarrow & & \downarrow & & \Vert \\
K_g & \rightarrowtail & K_{hg} & \longrightarrow & K_h & \twoheadrightarrow & Y & \rightarrowtail & L_g & \longrightarrow & L_s & \twoheadrightarrow & L_h
\end{array}
$$

$$
\begin{tikzcd}
K_f \arrow[r, tail] \arrow[d, equal] & K_r \arrow[r] \arrow[d, tail] & K_g \arrow[r, two heads] \arrow[d, tail] & X \arrow[r, tail] \arrow[d, tail] & L_f \arrow[r] \arrow[d, equal] & L_{gf} \arrow[r, two heads] \arrow[d] & L_g \arrow[d] \\
K_f \arrow[r, tail] \arrow[d] & A \arrow[r] \arrow[d, equal] & K_{hg} \arrow[r, two heads] \arrow[d] & K_s \arrow[r, tail] \arrow[d] & L_f \arrow[r] \arrow[d] & F \arrow[r, two heads] \arrow[d, equal] & L_s \arrow[d] \\
K_r \arrow[r, tail] \arrow[d] & A \arrow[r] \arrow[d] & K_h \arrow[r, two heads] \arrow[d, equal] & L_r \arrow[r, tail] \arrow[d, two heads] & L_{gf} \arrow[r] \arrow[d, two heads] & F \arrow[r, two heads] \arrow[d, two heads] & L_h \arrow[d, equal] \\
K_g \arrow[r, tail] & K_{hg} \arrow[r] & K_h \arrow[r, two heads] & Y \arrow[r, tail] & L_g \arrow[r] & L_s \arrow[r, two heads] & L_h
\end{tikzcd}
$$


K_f K_r K_g X L_f L_{gf} L_g
K_f A K_{hg} K_s L_f F L_s
K_r A K_h L_r L_{gf} F L_h
K_g K_{hg} K_h Y L_g L_s L_h

Thank you!