Noncommutative tensor triangular geometry and cohomological support varieties

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Big picture

Goal: Study monoidal triangulated categories (e.g. those arising from representation theory) via their inherent geometry.

Support varieties

Support varieties play one of the primary motivations for this theory.

Origins:

- Quillen (1971).
- Carlson (1983).

Much of the theory developed in this context extends to arbitrary categories of representations of Hopf algebras, tensor categories, and triangulated categories.

Yoneda product

In an abelian category:

$$\mathrm{Ext}^i(A,B) \times \mathrm{Ext}^j(C,A) \to \mathrm{Ext}^{i+j}(C,B)$$

by sending

$$0 \to B \to M_0 \to \ldots \to M_{i-1} \to A \to 0$$
$$\times$$
$$0 \to A \to N_0 \to \ldots \to N_{j-1} \to C \to 0$$
$$\mapsto$$

Yoneda product

$$
\begin{array}{c}
\qquad\qquad\qquad A \\
\qquad\qquad\quad \nearrow \quad \searrow \\
0 \to B \to M_0 \to \dots \to M_{i-1} \longrightarrow N_0 \to \dots \to N_{j-1} \to C \to 0
\end{array}
$$

$$\operatorname{Ext}^\bullet(A, A) \times \operatorname{Ext}^\bullet(A, A) \to \operatorname{Ext}^\bullet(A, A)$$

$$\operatorname{Ext}^\bullet(B, B) \circlearrowright \operatorname{Ext}^\bullet(A, B) \circlearrowleft \operatorname{Ext}^\bullet(A, A)$$

Monoidal categories

A monoidal category consists of a category C with a product $\otimes : \mathrm{C} \times \mathrm{C} \to \mathrm{C}$ and a unit 1 such that

- $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
- $A \otimes 1 \cong A \cong 1 \otimes A$.

Finite tensor categories

A monoidal category $(\mathrm{C}, \otimes, 1)$ is a finite tensor category if it is an abelian $\mathbb{k}$-linear monoidal category such that

- $- \otimes -$ is bilinear on spaces of morphisms;
- every object has finite length;
- $\mathrm{Hom}(A, B)$ is finite-dimensional;
- 1 is simple;
- there are enough projectives;
- every object is dualizable.

Think: $\mathrm{Rep}(H)$.

Cohomology ring of the unit

Duals $\Rightarrow - \otimes -$ is biexact.
There are two ring homomorphisms

$$\mathrm{Ext}^{\bullet}(1, 1) \to \mathrm{Ext}^{\bullet}(A, A)$$

defined by

$$0 \to 1 \to M_0 \to \ldots \to M_{i-1} \to 1 \to 0$$

$$\mapsto$$

$$0 \to 1 \otimes A \to M_0 \otimes A \to \ldots \to M_{i-1} \otimes A \to 1 \otimes A \to 0,$$

and

$$0 \to A \otimes 1 \to A \otimes M_0 \to \ldots \to A \otimes M_{i-1} \to A \otimes 1 \to 0.$$

Fix one for the sake of defining support varieties. Let's use $- \otimes A$.

Cohomology ring of the unit

Now define

$$H^\bullet(\mathrm{C}) = \begin{cases} \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(1,1) & \mathrm{char}(\Bbbk) = 2 \\ \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^{2i}(1,1) & \mathrm{char}(\Bbbk) \neq 2 \end{cases}$$

Denote $I(A,B)$ the annihilator of $\mathrm{Ext}^\bullet(A,B) \in H^\bullet(\mathrm{C})$.

Definition

The support variety corresponding to A and B is

$$W(A,B) = \{\mathfrak{p} \in \mathrm{Proj}\, H^\bullet(\mathrm{C}) : I(A,B) \subseteq \mathfrak{p}\}.$$

For $A = B$, we denote $W(A) := W(A,A)$.

Finite generation conditions

We say C satisfies (fg) if

- $H^\bullet(\mathrm{C})$ is a finitely-generated algebra.
- Each $\mathrm{Ext}^\bullet(A, B)$ is a finitely-generated $H^\bullet(\mathrm{C})$-module.

History of (fg):

- (fg) for finite groups: Golod (1959), Venkov (1959), Evens (1961).
- (fg) for small quantum groups: Ginzburg-Kumar (1993), conditions broadened in Bendel-Nakano-Parshall-Pillen (2014).
- (fg) for finite group schemes: Friedlander-Suslin (1997).
- Etingof-Ostrik (2004): conjectured for all finite tensor categories.
- (fg) for finite-dimensional pointed Hopf algebras with abelian group of grouplikes: Andruskiewitsch-Angiono-Pevtsova-Witherspoon (2020).

Projective objects

For P projective, $I(P) = H^{\bullet}(\mathrm{C})_+$, so $\Rightarrow W(P) = \varnothing$.In fact, if C satisfies (fg), then

$$W(A) = \varnothing \Leftrightarrow A \text{ is projective.}$$

The stable category

Want to “set projectives equal to 0”:

- Let $\mathrm{PHom}_{\mathrm{C}}(A, B)$ be morphisms $A \to B$ which factor through some projective object.
- Then st(C) is defined as the category where
  1. Objects: same as in C.
  2. Morphisms $A \to B$: $\mathrm{Hom}_{\mathrm{C}}(A, B) / \mathrm{PHom}_{\mathrm{C}}(A, B)$.

In st(C), $A \cong 0 \Leftrightarrow A$ is projective in C. This is still a monoidal category:

$$\mathrm{Hom}(P \otimes A, -) \cong \mathrm{Hom}(P, - \otimes A^*),$$

likewise for $A \otimes P$.

Triangulated structure on the stable category

st(C) is no longer abelian, but it is triangulated:

- $\Sigma = \Omega^{-1}$, where $\Omega(A)$ is the kernel of $P_0 \to A$ in a projective resolution of A.
- Triangles arise from short exact sequences of C.

The monoidal product is exact. Hence, st(C) is an example of a monoidal triangulated category.

The tensor product property

Straightforward: $W(A \otimes B) \subseteq W(A)$. If C (or just st(C)) is braided, then $W(A \otimes B) \subseteq W(A) \cap W(B)$.

Definition

We say W satisfies the tensor product property (TPP) if $W(A \otimes B) = W(A) \cap W(B)$.

- TPP for finite groups: Carlson (1983).
- TPP for finite group schemes: Friedlander-Pevtsova (2007).
- Counterexamples for noncocommutative Hopf algebras: Benson-Witherspoon (2014) and Plavnik-Witherspoon (2018).
- TPP for small quantum groups of Borel subalgebras in type A: Negron-Pevtsova (2020).

Thick ideals

A triangulated subcategory I of a monoidal triangulated category K is a thick ideal if

- $A \oplus B \in \mathrm{I}$ implies A or B in I.
- $A \in \mathrm{I}$ implies $A \otimes B$ and $B \otimes A \in \mathrm{I}$ for all $B \in \mathrm{K}$.

Thick ideals

Big picture:

- Goal 1: to understand a tensor category completely, understand all indecomposable objects and how they tensor together.
- In some cases, goal 1 is possible. In many cases, it's too hard! So we modify our goal.
- Goal 2: classify the thick ideals.

Thick ideal classification problems

Thick ideal classification problems originate in work of Hopkins (1985) and Neeman (1992).
A few of the central motivating results for tensor triangular geometry:

1. Thomason (1997): classified thick ideals of $D^{\mathrm{perf}}(X)$. Uses the geometry of X.
2. Friedlander-Pevtsova (2007): classified thick ideals of $\mathrm{stmod}(\Bbbk G)$. Uses geometry of $\mathrm{Proj}\, H^\bullet(G, \Bbbk)$.

Tensor triangular geometry

The classifications (1) and (2) above were united in the work of Paul Balmer.

- Recall that an ideal $\mathfrak{p}$ of a commutative ring is prime if $a \cdot b \in \mathfrak{p} \Rightarrow a$ or $b \in \mathfrak{p}$.
- For a ring R, the collection of prime ideals $\operatorname{Spec} R$ is a geometric space.
- Balmer defines P a thick ideal of a braided monoidal triangulated category K to be prime if $A \otimes B \in \mathsf{P}$ implies A or $B \in \mathsf{P}$.

Tensor triangular geometry

The collection of prime ideals is called the Balmer spectrum, denoted $\operatorname{Spc} \mathsf{K}$.

Theorem (Balmer)

The Balmer spectrum of K *is the universal final support.*

1. *If* X *is a topologically Noetherian scheme,* $\operatorname{Spc} D^{\text{perf}}(X) \cong X$.
2. *If* $\Bbbk G$ *is a finite group scheme, then* $\operatorname{Spc} \operatorname{stmod}(\Bbbk G) \cong \operatorname{Proj} H^\bullet(G, \Bbbk)$.

Tensor triangular geometry

Since 2005, Balmer's ideas have found wide applicability. A few results:

- Balmer (2010), Balmer-Sanders (2017).
- Matsui-Takahashi (2017).
- Boe-Kujawa-Nakano (2014, 2017).

Noncommutativity

Many examples of monoidal triangulated categories we might care about are not braided. For example:

- Let A be a noncommutative ring, whose enveloping algebra has finite global dimension. $D^b(\text{bimod}(A))$ is a n.c. monoidal triangulated category under $- \otimes_A^L -$.
- Let H be a non-quasitriangular Hopf algebra. Then stmod(H) is a n.c. monoidal triangulated category under $- \otimes_{\Bbbk} -$.

Noncommutative prime ideals

Let K be a monoidal triangulated category (e.g. st(C)).

- $\mathsf{I} \subset \mathsf{K}$ is a thick left ideal if it is triangulated, closed under direct summands, and $\mathsf{K} \otimes \mathsf{I} = \mathsf{I}$.
- A thick left ideal I is a thick ideal if $\mathsf{I} \otimes \mathsf{K} = \mathsf{I}$.
- A thick ideal P is prime if

$$\mathsf{I} \otimes \mathsf{J} \subseteq \mathsf{P} \Rightarrow \mathsf{I} \text{ or } \mathsf{J} \subseteq \mathsf{P}.$$

- Equivalent: $A \otimes \mathsf{K} \otimes B \subseteq \mathsf{P} \Rightarrow A$ or $B \in \mathsf{P}$.
- If P satisfies $A \otimes B \in \mathsf{P}$ implies A or $B \in \mathsf{P}$, then we say it is completely prime.

Noncommutative Balmer spectra

Spc(K) is the collection of prime ideals. Closed sets of Spc K:

$$V(\mathcal{S}) = \{\mathsf{P} \in \operatorname{Spc} \mathsf{K} : \mathcal{S} \cap \mathsf{P} = \varnothing\}.$$

Restricting to objects, we have a map

$$V : \mathsf{K} \to \{\text{closed sets in } \operatorname{Spc} \mathsf{K}\}$$

$$A \mapsto V(A) = \{\mathsf{P} \in \operatorname{Spc} \mathsf{K} : A \notin \mathsf{P}\}.$$

Define the map Φ_V by

$$\Phi_V(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} V(A)$$

for any subset $\mathcal{S}$ of K.

The maps V and Φ_V

Some properties of the maps V and Φ_V:

1. $V(0) = \emptyset$, $V(1) = \operatorname{Spc} \mathsf{K}$;
2. $V(A \oplus B) = V(A) \cup V(B)$;
3. $V(\Sigma A) = V(A)$;
4. If $A \to B \to C \to \Sigma A$ is a distinguished triangle, then $V(A) \subset V(B) \cup V(C)$;
5. $\Phi_V(\mathsf{I} \otimes \mathsf{J}) = \Phi_V(\mathsf{I}) \cap \Phi_V(\mathsf{J})$.

Support data

Definition

For a monoidal triangulated category K*, a weak support datum is a map* $\sigma : \mathsf{K} \to$ *closed sets in* X*, such that*

1. $\sigma(0) = \emptyset, \sigma(1) = X$;
2. $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$;
3. $\sigma(\Sigma A) = \sigma(A)$;
4. *If* $A \to B \to C \to \Sigma A$ *is a distinguished triangle, then* $\sigma(A) \subset \sigma(B) \cup \sigma(C)$;
5. $\Phi_\sigma(\mathsf{I} \otimes \mathsf{J}) = \Phi_\sigma(\mathsf{I}) \cap \Phi_\sigma(\mathsf{J})$.

The final support datum

Theorem

Let $\sigma : \mathsf{K} \to X$ be a weak support datum such that $\Phi_\sigma(\langle A \rangle)$ is closed for each object A. Then there is a unique continuous map $f : X \to \operatorname{Spc} \mathsf{K}$ with $f^{-1}(V(A)) = \Phi_\sigma(\langle A \rangle)$.

$$\begin{array}{ccc} & \mathsf{K} & \\ {}^{\sigma}\swarrow & & \searrow^{V} \\ X & \xrightarrow{\quad f \quad} & \operatorname{Spc} \mathsf{K} \end{array}$$

$$x \longmapsto \{A \in \mathsf{K} : x \notin \Phi_\sigma(\langle A \rangle)\}$$

The final support datum

Slogan / summary:

- "The Balmer spectrum is the universal way to assign closed subsets of a topological space to a monoidal triangulated category, in a way that respects the homological properties and tensor structure."

Compact objects

In order to use the tools of localization and colocalization functors (whose existence is due to Brown representability, see Keller (1994) and Neeman (1996)), we need to work in the context of compactly generated categories.

- C compact means $\mathrm{Hom}(C, -)$ commutes with arbitrary coproducts.
- For stable module categories, C compact $\Leftrightarrow$ isomorphic to a finite-dimensional module.

Compactly generated monoidal triangulated categories

Definition

A monoidal triangulated category K is said to be **compactly generated** if:

1. K contains set-indexed coproducts;
2. K is generated as a localizing subcategory by its collection of compact objects;
3. All compact objects of K are rigid;
4. The tensor product of compact objects is compact;
5. 1 is compact.

Faithfulness and realization conditions

We will be interested in support data that satisfy two additional conditions:

- The faithfulness property: $\Phi_\sigma(\langle M \rangle) = \varnothing$ if and only if $M = 0$, $\forall M \in \mathrm{K}$.
- The realization property: For any closed set S there exists $M \in \mathrm{K}^c$ such that $\Phi_\sigma(\langle M \rangle) = S$.

Classification of thick ideals and the Balmer spectrum

Theorem (Nakano-V.-Yakimov)

Let K *be a compactly generated monoidal triangulated category and* $\sigma : \mathsf{K} \to X$ *be a weak support datum and* X *a Zariski space such that* $\Phi_\sigma(\langle C \rangle)$ *is closed for every compact object* C *and* σ *satisfies the faithfulness and realization properties. Then:*

1. *There is a bijective correspondence*

$$\mathsf{I} \longmapsto \Phi_\sigma(\mathsf{I})$$

$$\{\textit{thick ideals of } \mathsf{K}^c\} \longleftrightarrow \{\textit{specialization-closed sets in } X\}$$

$$\{M \in \mathsf{K}^c : \Phi_\sigma(\langle M \rangle) \subset S\} \longleftarrow\!\!\!\!\!\shortmid S$$

2. *The map* $f : X \to \operatorname{Spc} \mathsf{K}^c$ *is a homeomorphism.*

Smash coproduct of a group algebra with a group coordinate ring

Let:

- G and H finite groups with H acting on G by group automorphisms;
- $\mathbb{k}$ a field of positive characteristic dividing the order of G;
- A denote the Hopf algebra dual to the smash product $\mathbb{k}[G]\#\mathbb{k}H$.

As an algebra, $A = \mathbb{k}G \otimes \mathbb{k}[H]$, and has Hopf structure

$$\Delta(g \otimes p_x) = \sum_{y \in H} (g \otimes p_y) \otimes (y^{-1}.g \otimes p_{y^{-1}x}),$$

$$\epsilon(g \otimes p_x) = \delta_{x,1} \quad \text{and} \quad S(g \otimes p_x) = x^{-1}.(g^{-1}) \otimes p_{x^{-1}}.$$

Smash coproduct of a group algebra with a group coordinate ring

Theorem (Nakano-V.-Yakimov)

(a) *There exists a bijection between thick two-sided ideals of* $stmod(A)$ *and specialization closed sets of* $H\text{-Proj}(H^\bullet(A, \Bbbk))$.

(b) *There exists a homeomorphism* $f : H\text{-Proj}(H^\bullet(A, \Bbbk)) \to \mathrm{Spc}(stmod(A))$.

Negron-Pevtsova conjecture for small quantum Borels

Conjecture (Negron-Pevtsova, 2020)

The cohomological support maps for all small quantum Borel algebras associated to arbitrary complex simple Lie algebras and arbitrary choices of group-like elements posses the tensor product property.

Usual small quantum Borels $u_\zeta(\mathfrak{b})$:

- Generators $E_\alpha, K_\alpha^{\pm 1}$, $\alpha \in \Pi$.
- $K_\alpha K_\beta = K_\beta K_\alpha$;
- $K_\alpha^{-1} K_\alpha = 1 = K_\alpha K_\alpha^{-1}$;
- $K_\alpha E_\beta K_\alpha^{-1} = \zeta^{\langle \beta, \alpha \rangle} E_\beta$;
- $E_\alpha^\ell = 0$, $K_\alpha^\ell = 1$.
- Quantum Serre relations involving the E_α's.

Negron-Pevtsova conjecture for small quantum Borels

Theorem (Nakano-V.-Yakimov)

The Negron-Pevtsova conjecture is true type, if $\ell > h$, odd, and if $\mathfrak{g}$ is type G_2 then $3 \nmid \ell$.

$$\begin{array}{ccc} & \mathrm{K} = \mathrm{stmod}(u_\zeta(\mathfrak{b})) & \\ \swarrow^{W} & & \searrow^{V} \\ \mathrm{Proj}\, H^\bullet(\mathrm{K}) & & \mathrm{Spc}\, \mathrm{K} \end{array}$$

Negron-Pevtsova conjecture for small quantum Borels

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$$\begin{array}{ccc} & \mathsf{K} = \mathrm{stmod}(u_\zeta(\mathfrak{b})) & \\ \swarrow^{W} & & \searrow^{V} \\ \mathrm{Proj}\, H^\bullet(\mathsf{K}) & \xleftrightarrow[\cong]{} & \mathrm{Spc}\, \mathsf{K} \end{array}$$

Proof sketch

1. The irreducible representations for $u_\zeta(\mathfrak{b})$ are 1-dimensional, $\otimes$-invertible, and the action they induce on $H^\bullet(\mathsf{K})$ is trivial.
2. $H^\bullet(\mathsf{K})$ is finitely-generated (see Ginzburg-Kumar, 1993).
3. The map $\Phi_W(\mathsf{I}) := \bigcup_{A \in \mathsf{I}} W(A)$ defines a bijection between right ideals of K and specialization closed sets of $\mathrm{Proj}(H^\bullet(\mathsf{K}))$.
4. Every thick right ideal of K is two-sided.
5. There is a homeomorphism $\mathrm{Proj}\, H^\bullet(\mathsf{K}) \to \mathrm{Spc}\, \mathsf{K}$ which commutes with the support maps.
6. Every prime ideal is completely prime $\Rightarrow$ W has the tensor product property.

Step 3

To prove step 3, we use a classification theorem which was proven in previous work.

Theorem (Nakano-V.-Yakimov)

Let K *be a compactly generated* $M\Delta C$ *and* $\sigma : \mathsf{K} \to \mathcal{X}$ *be a quasi support datum for a Zariski space* X *such that* $\Phi_\sigma(\langle C \rangle)$ *is closed for every compact object* C*. Assume that* σ *satisfies the faithfulness and realization properties, and an additional technical assumption. Then we have bijections* Φ_σ *and* Θ_σ

$$\{\textit{thick right ideals of } \mathsf{K}^c\} \underset{\Theta_\sigma}{\overset{\Phi_\sigma}{\rightleftarrows}} \mathcal{X}_{sp}$$

are mutually inverse.

Step 3

Realization is the most involved part: if S is a closed set of $\operatorname{Proj} H^\bullet(\mathrm{K})$, then there exists an object A such that $\Phi_W(\langle A \rangle_r) = S$.
Using step 1, we prove that $\Phi_W(\langle A \rangle_r) = W(A)$. Then the realization property follows from classical support variety realization via the Hopf analogues of Carlson's L_ζ objects / Koszul objects.

Step 4

But then we are able to prove something stronger: in fact $\Phi_W(\langle A \rangle) = W(A)$. (Different version of) classification theorem $\Rightarrow$

$$\{\text{thick 2-sided ideals of K}\} \underset{\Theta_\sigma}{\overset{\Phi_\sigma}{\rightleftarrows}} \mathcal{X}_{sp}$$

But Φ_W was already a bijection at the level of right ideals– $\Rightarrow$ all right ideals are 2-sided.

Step 5

Since Φ_W is a (containment-preserving) bijection, we obtain

$$\Phi_W(\langle \mathsf{I} \otimes \mathsf{J} \rangle) = \Phi_W(\mathsf{I}) \cap \Phi_W(\mathsf{J})$$

since $\langle \mathsf{I} \otimes \mathsf{J} \rangle = \mathsf{I} \cap \mathsf{J}$ (this uses the fact that all thick ideals are semiprime, using duals). This gives the map

$$f : \operatorname{Proj} H^\bullet(\mathsf{K}) \to \operatorname{Spc} \mathsf{K}$$

using the universal property of $\operatorname{Spc} \mathsf{K}$.

Step 6

All right ideals are two-sided $\Rightarrow$ all prime ideals are completely prime:

- Suppose $A \otimes B \in \mathrm{P}$.
- $A \otimes \langle B \rangle_r \subseteq \mathrm{P}$.
- $A \otimes \langle B \rangle \subseteq \mathrm{P}$.
- $A \otimes \mathrm{K} \otimes B \subseteq \mathrm{P}$.
- A or $B \in \mathrm{P}$.

This implies the Balmer support has the tensor product property:

$$V(A \otimes B) = V(A) \cap V(B).$$

Via the homeomorphism f, so does the cohomological support W.

Conclusion

Thank you for your time!