Pointwise Finite-Dimensional Representations of the Circle

or,

Pointwise Finite-Dimensional Representations of Type $\widetilde{\mathbb{A}}_{\mathbb{R}}$

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Outline

Up Next

1. Justification
2. Unifying Definition
3. Decomposition and Isomorphism Classes

Justification

Why study representations of the circle? Different kinds of representations of the circle have already appeared in several places. Here are some, in chronological order.

- D. Burghelea and T. K. Dey, *Topological Persistence for Circle-Valued Maps*, 2013
- K. Igusa and G. Todorov, *Continuous Frobenius Categories*, 2013
- F. Sala and O. Schiffmann, *Fock space representation of the circle quantum group*, 2019
- S. Guillermou, *Sheaves and symplectic geometry of cotangent bundles*, 2019

The various works above have already been cited in further work. However, each definition varies slightly from the others and we would like one definition that includes each of these.

Justification

But wait! There's more!

Recently, researchers have been interested in persistent homology indexed over infinite sets, indeed continuous sets.

- W. Crawley-Boevey, *Decomposition of pointwise finite-dimensional persistence modules*, 2015
- F. Chazal, W. Crawley-Boevey, and V. de Silva, *The observable structure of persistence modules*, 2016
- M. B. Botnan and W. Crawley-Boevey, *Decomposition of persistence modules*, `arXiv` 2018, published 2020
- K. Igusa, R., and G. Todorov, *Continuous Quivers of Type A (I) The Generalized BarCode Theorem*, 2019

Naturally we ask: what about continuous representations of type $\widetilde{\mathbb{A}}$?
$=$What about representations of the circle?

Up Next

Unifying Definition

To define representations of the circle we must first be precise about "the circle." First we'll choose an orientation of the circle.

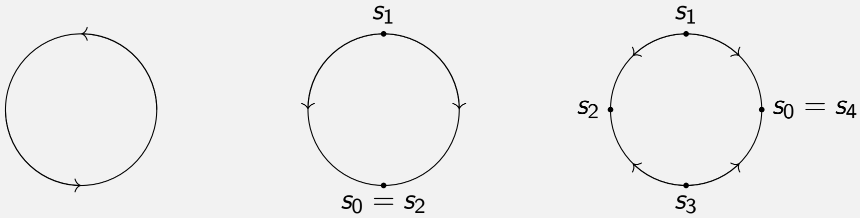
Definition (Orientation)

1. Choose an even number of points S on the circle (possibly 0).
2. If we chose $S = \emptyset$ the orientation is counterclockwise.
3. If we choose S nonempty, index the elements counterclockwise starting with s_0. We also index s_0 as s_n; this particularly helps with the case where $|S| = 2$.
4. If $S \neq \emptyset$ the orientation on the circle is the partial order where the even-indexed elements of S are sinks and the odd-indexed elements are sources.

This gives us a cyclic order when $|S| = 0$ and a partial order when $|S| \geq 2$. We denote the order by $\preceq$ and call this our **orientation**.

Unifying Definition

Examples (Orientations on the circle)



$|S| = 0$ $\qquad$ $|S| = 2$ $\qquad$ $|S| = 4$

Unifying Definition

Definition (Continuous Quiver of Type $\widetilde{\mathbb{A}}$)

Given S and $\preceq$, **a continuous quiver of type** $\widetilde{\mathbb{A}}$ is a category Q.

- The objects of Q are the points of $\mathbb{S}^1$.
- If $|S| = 0$ then for each pair of distinct points $x, y \in \mathbb{S}^1$:

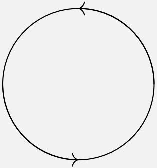
$$\mathrm{Hom}_Q(x, y) = \{g_{x,y} \circ \omega_x^n : n \in \mathbb{N}\} = \{\omega_y^n \circ g_{x,y} : n \in \mathbb{N}\},$$

  where $g_{x,y}$ is the unique morphism from x to y that travels counterclockwise less than one rotation and ω_x (ω_y) is the unique map from x (y) to itself that travels around $\mathbb{S}^1$ exactly once.
- If $|S| \geq 2$ then for each pair of distinct points $x, y \in \mathbb{S}^1$:

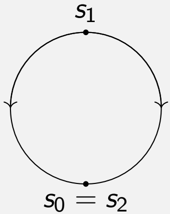
$$\mathrm{Hom}_Q(x, y) = \begin{cases} \{g^{\uparrow}_{x,y}, g^{\downarrow}_{x,y}\} & y = s_0, x = s_1, |S| = 2 \\ \{g_{x,y}\} & y \preceq x \\ \emptyset & y \not\preceq x. \end{cases}$$

Unifying Definition

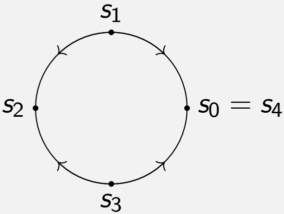
Examples (Orientations on the circle, now as categories)



$|S| = 0$



$|S| = 2$



$|S| = 4$

Unifying Definition

We denote by k-Vec and k-vec the categories of k-vector spaces and finite-dimensional k-vector spaces, for some field k.

Definition (Representation)

Let Q be a continuous quiver of type $\widetilde{\mathbb{A}}$. A **representation** of Q over a field k is a functor

$$V : Q \to k\text{-}Vec.$$

If V factors through k-vec we say V is **pointwise finite-dimensional**. We will write this as **pwf** for short.

There may be interesting representations that are not pointwise finite-dimensional, similar to those studied by F. Chazal, W. Crawley-Boevey, and V. de Silva.

However, for the rest of this presentation we'll restrict our attention to pwf representations.

Up Next

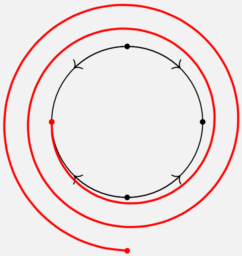
Isomorphism Classes

From now on, we will assume that we have fixed some field k.

"Definition" (Bar / String)

Let Q be a continuous quiver of type $\widetilde{\mathbb{A}}$ and V a representation of Q. If we can parameterize V by lifting to some bounded interval of $\mathbb{R}$ then we call V a **bar** or a **string**. A direct sum of bars is called a **bar code**.

Example (Bar / String)

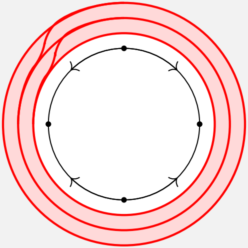


Isomorphism Classes

"Definition" (Jordan Cell / Band)

Let Q be a continuous quiver of type $\widetilde{\mathbb{A}}$ and V a representation of Q. If $V(g_{x,y})$ is an isomorphism for all $x, y \in \mathbb{S}^1$ and the "traveling around" map is not a nontrivial direct sum then we say V is a **Jordan cell** or **band**.

Example (Jordan Cell / Band)



If k is algebraically closed the "traveling around" map can be written as a Jordan block matrix.

Isomorphism Classes

Strings and bands are grouped into isomorphism classes in the same way as their discrete counterparts.

Theorem (Hanson-R. '20)

Let V and W be representations of a continuous quiver Q of type $\widetilde{\mathbb{A}}$.

1. *Suppose V and W are strings. Then $V \cong W$ if and only if they lift to the same interval of $\mathbb{R}$ modulo 2π.*
2. *Suppose V and W are bands; let $\widehat{V}$ and $\widehat{W}$ be the "traveling around" maps for V and W, respectively. Then $V \cong W$ if and only if there is a matrix A such that $\widehat{V} = A^{-1}\widehat{W}A$.*
3. *If V is a string and W is a band then $V \ncong W$.*

But! Are all pwf representations of continuous type $\widetilde{\mathbb{A}}$ a direct sum of strings and bands?

Decomposition

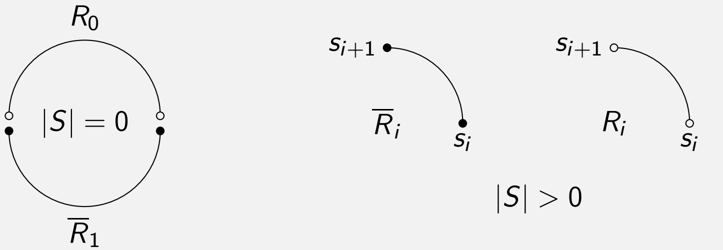
Fix a continuous quiver Q of type $\widetilde{\mathbb{A}}$.

We first consider the restriction a representation V to a region of $\mathbb{S}^1$. If $|S| \geq 2$, for each $0 \leq i < n$ let $\overline{R}_i$ be the closed region on $\mathbb{S}^1$ from s_i to s_{i+1}. (Recall that s_0 is also s_n.) The interior of the region (without s_i and s_{i+1}) is denoted R_i.

If $|S| = 0$ we just use the regions given by angles 0 to π and by angles π to 2π; call these regions $\overline{R}_0$ and $\overline{R}_1$, respectively. Then R_0 and R_1 are the open subintervals of these respective regions.

Decomposition

Examples of R_i and $\overline{R}_i$



Lemma (Hanson-R. '20)

Suppose W is a summand of V restricted to $\overline{R}_i$ and the support of W is contained in R_i. Then W is a summand of V.

Decomposition

Observation: Each region $\overline{R}_i$ is totally ordered.

Theorem (Crawley-Boevey '15)

A pwf representation of a totally-ordered set decomposes into interval indecomposables.

If we apply this to our restriction of V to each region $\overline{R}_i$ then we obtain a barcode W_i (which may be 0) where each bar of W_i has support in R_i. Now we have

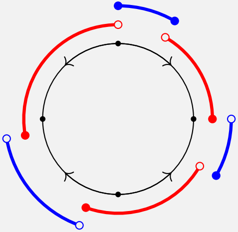
$$V \cong V' \oplus \left(\bigoplus_{i=0}^{n-1} W_i \right).$$

"Definition"

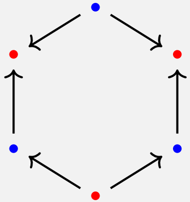
We say a pwf representation is **finitistic** if $V \cong V'$ above.

We now "lift" a finitistic representation V' to a finite-dimensional representation M_V of a type $\widetilde{\mathbb{A}}$ quiver Q_V. The discrete quiver is obtained by collapsing each piece in the partition to a point.

Example (Partition $\to$ Quiver)



Partition



Q_V

Decomposition

We then decompose M_V and push it back down.

Lemma (Hanson-R. '20)

- *A string or band representation of Q_V "pushes down" to a respective string or band representation of Q.*
- *If a representation of Q_V is indecomposable its "push-down" is indecomposable.*
- *Direct sums commute with "pushing down".*

We now have enough for the desired result.

Theorem (Hanson-R. '20)

- *A pointwise finite-dimensional representation of Q decomposes into a direct sum of string and finitely-many band representations.*
- *A pointwise finite-dimensional representation of Q is indecomposable if and only if it is either a string or band reprsentation.*

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- F. Chazal, W. Crawley-Boevey, and V. de Silva, *The observable structure of persistence modules*, Homology, Homotopy and Appl. **18** (2016), no. 2, 247–265, DOI: 10.4310/HHA.2016.v18.n2.a14
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