How to compute in free abelian categories

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Theorem

Let A be a commutative $\mathbb{Q}$-algebra, $a, b \in A$ such that

$$a^3 b + a = b^3 a + b = a^2 b^2 + 1 = 0.$$

Then $1 = 0$ in A.

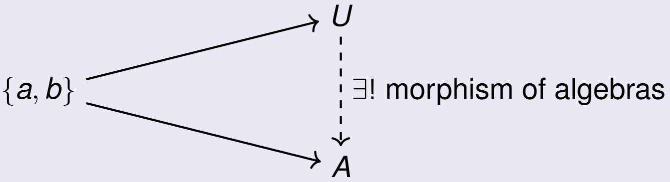
Proof strategy

1. Construct the universal commutative algebra U which satisfies the premises of the theorem.
2. Show that the conclusion $1 = 0$ holds for U.

1. Construct the universal commutative algebra U which satisfies the premises of the theorem.

The universal algebra which satisfies the premises.

- a commutative algebra U
- a map $\{a, b\} \to U$ with $a^3b + a = b^3a + b = a^2b^2 + 1 = 0$.
- For every commutative algebra A and map $\{a, b\} \to A$ whose images in A also satisfy that relation:



$$U := \mathbb{Q}[a, b]/\langle a^3b + a, b^3a + b, a^2b^2 + 1\rangle$$

2. Show that the conclusion $1 = 0$ holds for U. (Gröbner basis)

Q: Can we apply a similiar proof strategy for the Snake lemma?

Snake lemma

Suppose given the following commutative diagram with exact rows in an abelian category:

$$\begin{array}{ccccccc} & & A & \xrightarrow{a} & B & \longrightarrow & \mathrm{Cok}(a) & \longrightarrow & 0 \\ & & \big\downarrow e & & \big\downarrow b & & \big\downarrow d & & \\ 0 & \longrightarrow & \mathrm{Ker}(c) & \longrightarrow & C & \xrightarrow{c} & D & & \end{array}$$

Then we have an exact sequence

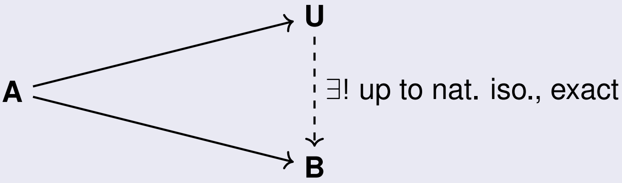
$$\mathrm{Ker}(e) \longrightarrow \mathrm{Ker}(b) \longrightarrow \mathrm{Ker}(d) \xrightarrow{\partial} \mathrm{Cok}(e) \longrightarrow \mathrm{Cok}(b) \longrightarrow \mathrm{Cok}(d)$$

Free abelian categories

Let **A** be an additive category.

Data of the free abelian category

- an abelian category **U**
- an additive functor **A** $\to$ **U**
- For every additive functor **A** $\to$ **B** into an abelian category:

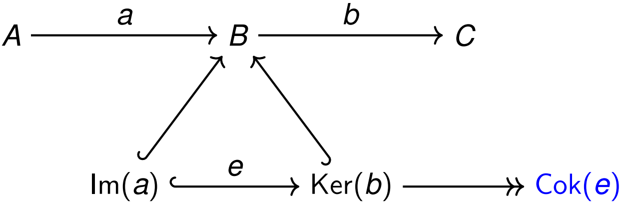


How to construct free abelian categories

- Existence: Peter Freyd
- Explicit construction: Murray Adelman

Homology of sequences

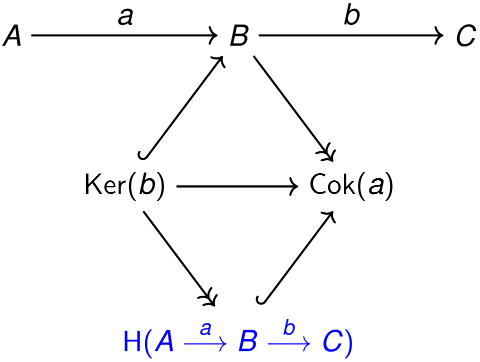
How to compute homology in an abelian category:



This construction only makes sense if we have $a \cdot b = 0$.

Homology of composite morphisms

How to compute homology in an abelian category:



This construction makes sense even if we don't have $a \cdot b = 0$.

Adelman's observation

Equipping an additive category with "homologies" makes it abelian.

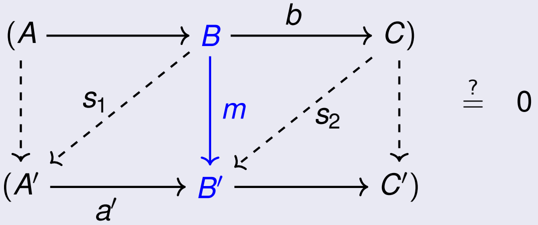
Adelman category

Let **A** be additive.

Adelman category: data structures

The Adelman category Adel(**A**) is given by the following data:

- An object in Adel(**A**) is a composite morphism $(A \xrightarrow{a} B \xrightarrow{b} C)$ in **A**.
- Morphisms:

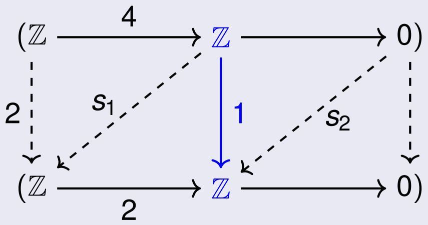


$$s_1 \cdot a' + b \cdot s_2 = m$$

The Adelman category is additive.

Some examples of morphisms

$\mathbf{A}$ = Category of free $\mathbb{Z}$-modules



A non-zero morphism in $\mathrm{Adel}(\mathbf{A})$.

This reminds us of a presentation of the morphism $\mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by free modules.

Some examples of morphisms

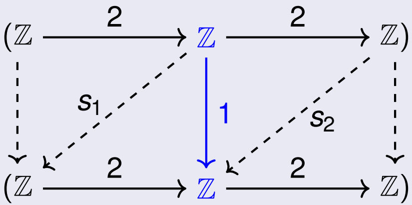
$\mathbf{A}$ = Category of free $\mathbb{Z}$-modules

$$
\begin{array}{ccccc}
(0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} \\
(0 & \longrightarrow & \mathbb{Z} & \xrightarrow[4]{} & \mathbb{Z})
\end{array}
$$

A non-zero morphism in $\mathrm{Adel}(\mathbf{A})$.

Some examples of morphisms

A $=$ Category of free $\mathbb{Z}$-modules



A non-zero morphism in $\mathrm{Adel}(\mathbf{A})$.

In particular, $(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z})$ is not isomorphic to the zero object.

Features of Adel($\mathbf{A}$)

Canonical embedding

$$\mathbf{A} \longrightarrow \mathrm{Adel}(\mathbf{A})$$
$$A \mapsto (0 \to A \to 0)$$

Duality

$$\mathrm{Adel}(\mathbf{A}) \xrightarrow{\sim} \mathrm{Adel}(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}}$$
$$(A \xrightarrow{a} B \xrightarrow{b} C) \mapsto (C \xrightarrow{b^{\mathrm{op}}} B \xrightarrow{a^{\mathrm{op}}} A)$$

$\rightsquigarrow$ if Adel($\mathbf{A}$) has cokernels, then it also has kernels

How to compute (co)kernels

$$
\begin{array}{ccccclc}
(0 & \longrightarrow & B & \xrightarrow{\;m\;} & B') & \Big\}= \mathrm{Ker}(\alpha) \\
\downarrow & & \downarrow & & \downarrow & \\
(0 & \longrightarrow & B & \longrightarrow & 0) & \\
\downarrow & & \downarrow m & & \downarrow & \Big\} =: \alpha \\
(0 & \longrightarrow & B' & \longrightarrow & 0) & \\
\downarrow & & \downarrow & & \downarrow & \\
(B & \xrightarrow{\;m\;} & B' & \longrightarrow & 0) & \Big\}= \mathrm{Cok}(\alpha)
\end{array}
$$

How to compute (co)kernels

$$
\begin{array}{ccccc}
(A \oplus A' & \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & \mathrm{id}_{A'} \end{pmatrix}} & B \oplus A' & \xrightarrow{\begin{pmatrix} b & m \\ 0 & a' \end{pmatrix}} & C \oplus B') \; \} = \mathrm{Ker}(\alpha) \\
\downarrow & & \downarrow & & \downarrow \\
(A & \xrightarrow{a} & B & \xrightarrow{b} & C) \\
\downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\
(A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C') \\
\downarrow & & \downarrow & & \downarrow \\
(A' \oplus B & \xrightarrow{\begin{pmatrix} a' & 0 \\ m & b \end{pmatrix}} & B' \oplus C & \xrightarrow{\begin{pmatrix} b' & 0 \\ 0 & \mathrm{id}_C \end{pmatrix}} & C' \oplus C) \; \} = \mathrm{Cok}(\alpha)
\end{array}
$$

$$
\left.\begin{array}{l} (A \xrightarrow{a} B \xrightarrow{b} C) \\ (A' \xrightarrow{a'} B' \xrightarrow{b'} C') \end{array}\right\} =: \alpha
$$

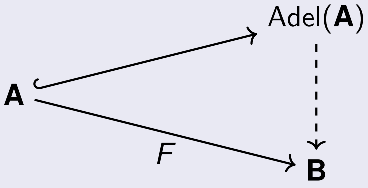
Adelman's theorem

Let $\mathbf{A}$ be an additive category.

Theorem

$\mathbf{A} \hookrightarrow \mathrm{Adel}(\mathbf{A})$ is the free abelian category of $\mathbf{A}$.

Its universal property is given as follows:



$$(A \xrightarrow{a} {\color{blue}B} \xrightarrow{b} C) \longmapsto \mathrm{H}(FA \xrightarrow{Fa} FB \xrightarrow{Fb} FC)$$

Computability of Adelman categories

$\mathrm{Adel}(\mathbf{A})$ provides a **computable** model of the free abelian category whenever we can solve 2-sided linear systems in $\mathbf{A}$.

1. Free abelian categories
2. A computational proof of the Snake lemma

Snake lemma

Suppose given the following commutative diagram with exact rows in an abelian category:

$$\begin{array}{ccccccc} & & A & \xrightarrow{a} & B & \longrightarrow & \mathrm{Cok}(a) & \longrightarrow & 0 \\ & & \downarrow e & & \downarrow b & & \downarrow d & & \\ 0 & \longrightarrow & \mathrm{Ker}(c) & \longrightarrow & C & \xrightarrow{c} & D & & \end{array}$$

Then we have an exact sequence

$$\mathrm{Ker}(e) \longrightarrow \mathrm{Ker}(b) \longrightarrow \mathrm{Ker}(d) \xrightarrow{\partial} \mathrm{Cok}(e) \longrightarrow \mathrm{Cok}(b) \longrightarrow \mathrm{Cok}(d)$$

1. Construct the universal abelian category **U** which satisfies the premises of the Snake lemma.
2. Show that the conclusion holds for **U**.

Snake lemma (rephrased)

Suppose given three consecutive morphisms in an abelian category

$$A \xrightarrow{\;a\;} B \xrightarrow{\;b\;} C \xrightarrow{\;c\;} D$$

such that

$$a \cdot b \cdot c = 0.$$

Then we have an exact sequence

$$\mathrm{Ker}(e) \longrightarrow \mathrm{Ker}(b) \longrightarrow \mathrm{Ker}(d) \xrightarrow{\partial} \mathrm{Cok}(e) \longrightarrow \mathrm{Cok}(b) \longrightarrow \mathrm{Cok}(d)$$

where d and e are defined by the following diagram:

$$\begin{array}{ccccccc} & & A & \xrightarrow{\;a\;} & B & \longrightarrow & \mathrm{Cok}(a) & \longrightarrow 0 \\ & & \downarrow e & & \downarrow b & & \downarrow d & \\ 0 & \longrightarrow & \mathrm{Ker}(c) & \longrightarrow & C & \xrightarrow{\;c\;} & D & \end{array}$$

Construct the universal abelian category **U** which satisfies the premises of the Snake lemma.

We set

$$\mathbf{U} := \mathrm{Adel}(\mathbf{A})$$

where **A** is the additive closure of the category freely generated by three consecutive morphisms $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ such that $a \cdot b \cdot c = 0$.

Example of a morphism in **A**

$$A \oplus B \xrightarrow{\begin{pmatrix} \mathrm{id}_A & a \cdot b \\ 0 & -3b \end{pmatrix}} A \oplus C$$

Remark

For every abelian category **B**, an additive functor $\mathbf{A} \to \mathbf{B}$ is determined (up to nat. iso.) by three consecutive morphisms in **B** whose composition yields 0.

Proof of the Snake lemma

Construct the universal abelian category $\mathbf{U}$ which satisfies the premises of the Snake lemma.

We set

$$\mathbf{U} := \mathrm{Adel}(\mathbf{A})$$

where $\mathbf{A}$ is the additive closure of the category freely generated by three consecutive morphisms $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ such that $a \cdot b \cdot c = 0$.

Show that the conclusion holds for $\mathbf{U}$.

Software demo!

https://github.com/sebastianpos/Adelman.jl

References I

- Murray Adelman, *Abelian categories over additive ones*, J. Pure Appl. Algebra **3** (1973), 103–117. MR 0318265
- Sebastian Posur, *A constructive approach to Freyd categories*, Applied Categorical Structures (2021), (`10.1007/s10485-020-09612-y`).
- Sebastian Posur, *On free abelian categories for theorem proving*, (`arXiv: 2103.08379`), 2021.