Hochschild cohomology of general twisted tensor products

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Roadmap

1 Introduction
2. Hochschild cohomology
3 Twisted tensor products.
4. Results.
5 Applications.

1. Introduction

A unital associative algebra over k field

$HH^*(A)$. The Hochschild cohomology encodes infinitesimal information

$HH^0(A) \cong Z(A)$. the center of A

$HH^1(A) \cong \operatorname{OutDer}(A)$ the outer derivations of A

$HH^2(A)$ the "important" infinitesimal deformations of A.

$A \otimes_\tau B$: Čap, Schichl, Vanžura: whenever an algebra has an underlying vector space given by the tensor product of two subalgebras, then it is iso. to a twisted tensor product.

Originally this came from non-commutative geometry.

Nowadays, this has applications in: operator algebras, algebraic topology, quantum symmetries.

2. Hochschild cohomology

Def: The Hochschild cohomology is. $HH^n(A) := \operatorname{Ext}^n_{A^e}(A,A)$, $HH^*(A) := \bigoplus_{n\in\mathbb{N}} \operatorname{Ext}^n_{A^e}(A,A)$.

$A^e = A \otimes A^{op}$

Def: Bar resolution: consider $A^{\otimes(n+2)}$ as an A^e-mod, $n \in \mathbb{N}$, and:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A\otimes A \xrightarrow{m_A} A \longrightarrow 0$$

with:

$$d_n(a_0\otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \quad \text{for } a_j \in A.$$

Operations:

Cup product: $\quad \cup : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A)$

Gerstenhaber bracket: $\quad [-,-] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$

Structures:

$(HH^*(A), \cup)$ graded commutative algebra

$(HH^*(A), [-,-])$ graded Lie algebra

$(HH^*(A), \cup, [-,-])$ Gerstenhaber algebra.

3. Twisted tensor products.

Def: Given A, B, a twisting map $\tau : B\otimes A \longrightarrow A\otimes B$ is a bijective linear map satisfying:

$$\tau(1_B\otimes a) = a\otimes 1_B, \quad \tau(b\otimes 1_A) = 1_A\otimes b \quad \text{for all } a\in A,\ b\in B;$$

$$\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "m_B\otimes m_A"'] & B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] \arrow[d, phantom, "\circlearrowright"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
B\otimes A \arrow[r, "\tau"] & A\otimes B & A\otimes A\otimes B\otimes B \arrow[l, "m_A\otimes m_B"']
\end{tikzcd}$$

multiplying   twisting   twisting   multiply

The twisted tensor algebra $A\otimes_\tau B$ is $A\otimes B$ with multiplication:

$$m_{A\otimes_\tau B} : A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B$$

We want to understand $HH^\cdot(A \otimes_\tau B)$.

Def: Let M, N be A-bim, B-bim respectively, there is a notion of compatibility with τ.

$$M \otimes B \longleftarrow B \otimes M$$
$$A \otimes N \longleftarrow N \otimes A$$

Under these compatibilities, we can give $M \otimes N$ a structure of $(A \otimes_\tau B)$-bimodule.

There is a notion of $P_\cdot \to A$, $Q_\cdot \to B$ resolutions being compatible with τ.

Theorem: [Shepler-Witherspoon] Under these compatibility conditions, $P_\cdot \to A$, $Q_\cdot \to B$, we can construct

$$P \otimes_\tau Q_\cdot \longrightarrow A \otimes_\tau B.$$

Examples:

1. The bar resolutions are always compatible for all $\tau: B \otimes A \longrightarrow A \otimes B$.
2. The Koszul resolutions of A, B are compatible with strongly graded τ.

4. Results.

[Negron-Witherspoon] [Grimley-Nguyen-Witherspoon] [Shepler-Witherspoon] [Volkov]

Thm: [KMOOW] Let $P_\cdot \to A$, $Q_\cdot \to B$ be projective bimodule resolutions, such that:

(i) $P_\cdot \otimes_\tau Q \longrightarrow A \otimes_\tau B$ is nice.

(ii) $\sigma : (P_\cdot \otimes_\tau Q_\cdot) \underset{A \otimes_\tau B}{\otimes} (P \otimes_\tau Q_\cdot) \longrightarrow (P_\cdot \otimes_A P_\cdot) \otimes_\tau (Q \otimes_B Q_\cdot)$ is nice.

Then we give an explicit formula for the Gerstenhaber bracket.

Prop: [KMOOW] This applies to:

1. The bar resolution.
2. The Koszul resolution for strongly graded τ

Our formulas are then applicable in full generality.

5. Applications.

The Jordan plane: $\quad k\langle x, y\rangle \quad \cong k[x] \otimes_\tau k[y]$ with $\tau(y \otimes x) = x \otimes y + x^2 \otimes 1$.

$(yx - xy - x^2)$

Thm: [KMOOW] We provide an explicit description of the Gerstenhaber algebra structure on

$$HH^*\left(\frac{k\langle x,y\rangle}{(yx-xy-x^2)}\right).$$

[Lopes-Solotar]

Thank you!

$$HH(A \otimes_{\mathbb{C}} B) \overset{?}{\cong} HH(A) \otimes_{\mathbb{C}}^{?} HH(B)$$