Generalized Gorenstein projective and flat modules

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Outline

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3. Gorenstein FP_n-projective modules
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Based on the following papers:

1. S. Estrada, A. Iacob, M. Perez. *“Model structures and relative Gorenstein flat modules and chain complexes”*, chapter in Contemporary Mathematics, Volume 751, ISBN 978-1-4704-4368-9, pages 135 176.

2. A. Iacob: *“Projectively coresolved Gorenstein flat and Ding projective modules”*, Communications in Algebra, 48(7): 2883 – 2893, 2020.

3. A. Iacob: *“Generalized Gorenstein modules”*, submitted.

4. D. Bravo, S. Estrada, A. Iacob. *“FP_n-injective and FP_n-flat covers and preenvelopes and Gorenstein AC-flat covers”*, Algebra Colloquium, 25(2), pages 319 - 334, 2018.

Definition

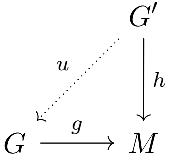
We say that a module $G \in Mod(R)$ is **Gorenstein projective** if there is an exact complex of projective modules $\mathbf{P} = \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \to \ldots$ such that $G = Z_0(P)$ and such that the complex stays exact when applying a functor $Hom(-, T)$, where T is any projective module (i.e. the complex $\ldots \to Hom(P_{-1}, T) \to Hom(P_0, T) \to Hom(P_1, T) \to \ldots$ is exact for any projective module T).

Any projective module P is Gorenstein projective $(0 \to P \xrightarrow{Id} P \to 0)$

Definition

We say that a module $M \in Mod(R)$ is **Gorenstein flat** if there is an exact complex of flat modules $\mathbf{F} = \ldots \to F_1 \to F_0 \to F_{-1} \to \ldots$ such that $M = Z_0(F)$ and such that the complex stays exact when applying a functor $A \otimes -$, where A is any injective module (i.e. the complex $\ldots \to A \otimes F_1 \to A \otimes F_0 \to A \otimes F_{-1} \to \ldots$ is exact for any injective module A).

A homomorphism $\phi : G \to M$ is a *Gorenstein projective precover* of M if G is Gorenstein projective and if for any Gorenstein projective module G' and any $\phi' \in Hom(G', M)$ there exists $u \in Hom(G', G)$ such that $\phi' = \phi u$.



A precover $g : G \to M$ is said to be a *cover* if any homomorphism $u : G \to G$ such that $gu = g$, is an isomorphism.

A Gorenstein projective resolution of a module M is a complex

$$\ldots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \to 0$$

such that $G_0 \to M$ and each $G_i \to Ker(G_{i-1} \to G_{i-2})$ for $i \geq 1$ are Gorenstein projective precovers.

Open question: the existence of the Gorenstein projective resolutions.
Generalizations of the Gorenstein modules - the Ding projective modules

- The *Ding projective modules* are the cycles of the exact complexes of projective modules that remain exact when applying a functor $Hom(-, F)$, with F any flat module.

Open question: is the class of Ding projectives, $\mathcal{DP}$, precovering over any ring?

FP_n-injective and FP_n-flat modules

Definition

A module M is *n-finitely presented* (FP_n for short) if there exists an exact sequence $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated free. A module M is FP_∞ if and only if $M \in FP_n$ for all $n \geq 0$.

$FP_0 \supseteq FP_1 \supseteq \ldots \supseteq FP_n \supseteq FP_{n+1} \supseteq \ldots \supseteq FP_\infty$, with FP_0 the class of all finitely generated modules, and FP_1 the finitely presented modules.
A module M is FP_n*-injective* if $Ext^1_R(F, M) = 0$ for all $F \in FP_n$.
From the definition, we get the following ascending chain:

$$Inj = \mathcal{FI}_0 \subseteq \mathcal{FI}_1 \subseteq \cdots \subset \mathcal{FI}_\infty.$$

A module N is FP_n*-flat* if $Tor_1(F, N) = 0$ for all $F \in FP_n$.

From the definition, we get the following ascending chain:

$$Flat = \mathcal{FF}_0 = \mathcal{FF}_1 \subseteq \mathcal{FF}_2 \subseteq \cdots \subset \mathcal{FF}_\infty.$$

Definition

A module G is Gorenstein FP_n-projective if it a cycle in an exact complex of projective modules that remains exact when applying a functor $Hom(-, L)$ for any $L \in \mathcal{FF}_n$.
$\mathcal{GP}_n$ denotes the class of Gorenstein FP_n-projective modules.

We use $\mathcal{GP}_n$ to denote the class of Gorenstein $\mathcal{FP}_n$-projective modules.
- Since $\mathcal{FF}_1 = Flat$, $\mathcal{GP}_1 = \mathcal{DP}$ (the Ding projective modules).
- And $\mathcal{FF}_\infty = Level$, so $\mathcal{GP}_\infty = \mathcal{GP}_{ac}$ (the Gorenstein AC-projective modules.

By definition we have an ascending chain

$$\mathcal{GP}_\infty = \mathcal{GP}_{ac} \subseteq \cdots \subseteq \mathcal{GP}_2 \subseteq \mathcal{GP}_1 = \mathcal{DP} \subseteq \mathcal{GP}.$$

Main result for Gorenstein FP_n-projective modules:
Theorem A: Let R be any ring. For any $n \geq 2$, $\mathcal{GP}_n$ is a precovering class.

A sufficient condition for a class $\mathcal{C}$ be precovering is to be the left half of a complete cotorsion pair.
Recall $\mathcal{C}^{\perp} = \{M, Ext^1(C, M) = 0$, for all $C \in \mathcal{C}\}$
and ${}^{\perp}\mathcal{C} = \{L, Ext^1(L, C) = 0$, for all $C \in \mathcal{C}\}$
- A pair $(\mathcal{C}, \mathcal{L})$ is a *cotorsion pair* if $\mathcal{C}^{\perp} = \mathcal{L}$ and ${}^{\perp}\mathcal{L} = \mathcal{C}$.
- A cotorsion pair $(\mathcal{C}, \mathcal{L})$ is *complete* if for every M there are short exact sequences $0 \to L \to C \to M \to 0$ and $0 \to M \to L' \to C' \to 0$ with $C, C' \in \mathcal{C}$ and with $L, L' \in \mathcal{L}$.
A cotorsion pair $(\mathcal{C}, \mathcal{L})$ is *hereditary* if $Ext^i(C, L) = 0$ for any $C \in \mathcal{C}$, any $L \in \mathcal{L}$, all $i \geq 1$.
Examples: $(Proj, Mod)$, (Mod, Inj).

Known: for $n \geq 2$, $M \in \mathcal{FF}_n \Leftrightarrow M^+ \in \mathcal{FI}_n$ (where $M^+ = Hom_Z(M, Q/Z)$)
and $C \in \mathcal{FI}_n \Leftrightarrow C^+ \in \mathcal{FF}_n$.

So, for $n \geq 2$, $(\mathcal{FI}_n, \mathcal{FF}_n)$ is a duality pair in the sense of Bravo - Gillespie - Hovey.

Theorem

(Bravo - Gillespie - Hovey) Let R be a ring and suppose $(\mathcal{C}, \mathcal{D})$ is a duality pair such that $\mathcal{D}$ is closed under pure quotients. Let P be a complex of projective modules. Then $A \otimes P$ is exact for all $A \in \mathcal{C}$ if and only if $Hom(P, N)$ is exact for all $N \in \mathcal{D}$.

Proposition

A module M is Gorenstein FP_n-projective if and only if there is an exact complex of projective modules $P = \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \to \ldots$ such that $M = Z_0(P)$ and such that $A \otimes P$ is exact for all $A \in \mathcal{FI}_n$.

More general:

Definition

Let $\mathcal{B}$ be a fixed class of right R-modules. We say that a module M is projectively coresolved Gorenstein $\mathcal{B}$-flat if $M = Z_0(P)$ for some $B \otimes -$-acyclic and exact complex P of projective modules.

- $\mathcal{PGF}_{\mathcal{B}}$ denotes the class of projectively coresolved Gorenstein $\mathcal{B}$-flat modules.

Question: When is $\mathcal{PGF}_{\mathcal{B}}$ precovering?

- A class of modules $\mathcal{D}$ is *definable* if it is closed under direct products, direct limits and pure submodules.
(X is a pure submodule of Y if there is a pure short exact sequence

$$\rho \colon 0 \to X \to Y \to Y/X \to 0$$

i.e. an exac sequence such that the induced sequence

$$Hom_{\mathcal{G}}(L, \rho) \colon 0 \to Hom_{\mathcal{G}}(L, X) \to Hom_{\mathcal{G}}(L, Y) \to Hom_{\mathcal{G}}(L, X/Y) \to 0$$

in Ab is exact for every finitely presented module L).

- The definable closure of $\mathcal{B}$, $< \mathcal{B} >$, is the smallest definable class containing $\mathcal{B}$.
- An *elementary cogenerator* of a definable class $\mathcal{D}$ is a pure-injective module $D_0 \in \mathcal{D}$ such that every $D \in \mathcal{D}$ is a pure submodule of some product of copies of D_0.
Here, *pure-injective* means injective with respect to pure exact sequences.

Definition

We say that a class $\mathcal{B}$ is **semi-definable** if it is closed under products and contains an elementary cogenerator of its definable closure.

Theorem

(joint with Estrada and Perez) If $\mathcal{B}$ is a semi-definable class of right R-modules then $(\mathcal{PGF}_{\mathcal{B}}, \mathcal{PGF}_{\mathcal{B}}^{\perp})$ is a complete hereditary cotorsion pair. In particular, the class $\mathcal{PGF}_{\mathcal{B}}$ is precovering.

Since for any $n > 1$ the class of $\mathcal{FP}_n$-injective modules, $\mathcal{FI}_n$, is definable (so semi-definable also), and since $\mathcal{GP}_n = \mathcal{PGF}_{\mathcal{FI}_n}$, we obtain:

Theorem

(Theorem A) Let $n \geq 2$. The class of generalized Gorenstein FP_n-projective modules, $\mathcal{GP}_n$, is precovering.

Case $n = 1$

Lemma

$\mathcal{PGF} = \mathcal{DP} \bigcap \mathcal{GF}$.

Corollary

Over any ring R, $\mathcal{PGF} = \mathcal{DP}$ if and only if $\mathcal{DP} \subseteq \mathcal{GF}$.

Proposition

The Gorenstein flat dimension of a Ding projective module is either zero or infinite.

Proposition

The following are equivalent:

1. $\mathcal{DP} = \mathcal{PGF}$
2. *Every Ding projective module has finite Gorenstein flat dimension.*

Proposition

If R has finite left weak Gorenstein global dimension then $\mathcal{DP} = \mathcal{PGF}$.

Theorem

Let R be any ring. The following are equivalent:
(1) $\mathcal{DP} \subseteq \mathcal{GF}$.
(2) $\mathcal{DP} = \mathcal{PGF}$
(3) For any Ding projective module M, its character module, M^+, is Gorenstein injective.
(4) The class Inj^+ of all character modules of injective right R-modules, is contained in $\mathcal{DP}^\perp$.

Theorem

Let R be a right coherent ring. Then $\mathcal{DP} = \mathcal{PGF} = \mathcal{GP}_{ac}$

The coherence is a sufficient condition, but it is not a necessary condition on the ring. If R has finite global dimension (but it is not coherent) then $\mathcal{DP} = \mathcal{PGF}$.

Example. The ring

$$R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} & \mathbb{R} \\ 0 & 0 & \mathbb{Q} \end{bmatrix} / \begin{bmatrix} 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is noncoherent of finite global dimension.
So, $\mathcal{DP} = \mathcal{PGF}$ over R.

All the results in this section are joint with S. Estrada and M. Perez

Definition

Let $\mathcal{B}$ be a class of right R-modules. We say that a module $M \in Mod(R)$ is **Gorenstein $\mathcal{B}$-flat** if $M = Z_0(F)$ for some $(\mathcal{B} \otimes_R -)$-acyclic and exact complex F of flat modules.

1. Gorenstein flat modules are obtained when $\mathcal{B} = \mathcal{I}nj$.
   If $\mathcal{B} \supseteq \mathcal{I}nj$ then any Gorenstein $\mathcal{B}$-flat module is, in particular, a Gorenstein flat module.
2. Recall that a module $M \in Mod(R)$ is *of type* FP_∞ if there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

   with P_k finitely generated and projective for every $k \geq 0$.
   When $\mathcal{B} = \mathcal{FI}_\infty = \mathcal{AC} = (\mathcal{FP}_\infty)^\perp$ we obtain the class $\mathcal{GF}_{\mathrm{AC}}(R)$ of *Gorenstein AC-flat modules.*

Properties of Gorenstein AC-flat modules

1. $\mathcal{GF}_{\mathrm{AC}}$ is a precovering class over any ring R.
2. If $\mathcal{GF}_{\mathrm{AC}}$ is closed under extensions then $\mathcal{GF}_{\mathrm{AC}}(R)$ is a covering class.

Remark. Our new results show that Gorenstein AC-flat modules are always closed under extensions, and so the latter two properties hold for any ring R.

Lemma

Let $\mathcal{B}$ be a class of right R-modules. Then, the class $\mathcal{GF}_{\mathcal{B}}$ of Gorenstein $\mathcal{B}$-flat modules is a precovering class.

Proposition

If the class $\mathcal{GF}_{\mathcal{B}}$ of Gorenstein $\mathcal{B}$-flat modules is closed under extensions, then it is closed under taking kernels of epimorphisms and under direct limits. As a consequence, $\mathcal{GF}_{\mathcal{B}}$ is a covering class.

Proposition

If $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions, then the pair $(\mathcal{GF}_{\mathcal{B}}, \mathcal{GC}_{\mathcal{B}})$ is a complete and hereditary cotorsion pair in $Mod(R)$, where $\mathcal{GC}_{\mathcal{B}}$ be the right orthogonal class $\mathcal{GF}_{\mathcal{B}}^{\perp}$. .

Question: When is the class $\mathcal{GF}_{\mathcal{B}}$ closed under extensions?
We show that for any semi-definable class $\mathcal{B}$ we have
$\mathcal{GF}_{\mathcal{B}} = ^{\perp}(\mathcal{C} \bigcap \mathcal{PGF}_{\mathcal{B}}^{\perp})$, and so $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions.

We use:

Lemma

The following are equivalent for any R-module M and any class of right R-modules $\mathcal{B}$:
(a) M is Gorenstein $\mathcal{B}$-flat.
(b) $Tor_i(B, M) = 0$ for all $i \geq 1$ and $B \in \mathcal{B}$, and there exists an exact and $(B \otimes \ )$-acyclic sequence of modules $0 \to M \to F^0 \to F^1 \to \dots$ where each F^i is flat.
(c) There exists a short exact sequence of modules $0 \to M \to F \to G \to 0$ where F is flat and G is Gorenstein $\mathcal{B}$-flat.

Theorem

Let $\mathcal{B}$ be a semi-definable class of right R-modules. Then, the following conditions are equivalent for every $M \in \mathsf{Mod}(R)$:

(a) *M is Gorenstein $\mathcal{B}$-flat.*

(b) *There is a short exact sequence of modules*

$$0 \to F \to L \to M \to 0$$

with $F \in \mathcal{F}lat$ and $L \in \mathcal{PGF}_{\mathcal{B}}$, which is also $Hom_R(-, \mathcal{C})$-acyclic, for any cotorsion module C .

(c) *$Ext^1_R(M, C) = 0$ for every $C \in \mathcal{C} \cap \mathcal{PGF}_{\mathcal{B}}^{\perp}$.*

(d) *There is a short exact sequence of modules*

$$0 \to M \to F \to L \to 0$$

with $F \in \mathcal{F}lat$ and $L \in \mathcal{PGF}_{\mathcal{B}}$.

(a) *M is Gorenstein $\mathcal{B}$-flat.*
(b) *There is a short exact sequence of modules*

$$0 \to F \to L \to M \to 0$$

with $F \in \mathcal{F}lat$ and $L \in \mathcal{PGF}_{\mathcal{B}}$, which is also $Hom_R(-, \mathcal{C})$-acyclic, for any cotorsion module C .

Proof of (a) $\Rightarrow$ (b) $M = Z_0(F)$, F an acyclic complex of flat modules, that is $B \otimes -$ exact.

$(dw(Proj), (dwProj)^{\perp}$ is complete $\Rightarrow$ exact $0 \to G \to P \to F \to 0$, $P \in dw(Proj)$, $G \in (dwProj)^{\perp}$.

Then G is flat.

F and G are $\mathcal{B} \bigotimes -$ exact, so P is $\mathcal{B} \bigotimes -$ exact.

Exact sequence $0 \to Z_i G \to Z_i P \to Z_i F \to 0$ with $Z_i G$ flat, and $Z_i P \in \mathcal{PGF}_{\mathcal{B}}$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_jG & \xrightarrow{f} & Z_jP & \longrightarrow & Z_jF & \longrightarrow & 0 \\
 & & \downarrow h & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_j & \xrightarrow{g} & P_j & \longrightarrow & F_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{j-1}G & \longrightarrow & Z_{j-1}P & \longrightarrow & Z_{j-1}F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

If C is a cotorsion module, both g and h are C-injective, so f is also C-injective.

(b) *There is a short exact sequence of modules*

$$0 \to F \to L \to M \to 0$$

with $F \in \mathcal{F}lat$ *and* $L \in \mathcal{PGF}_{\mathcal{B}}$, *which is also* $Hom_R(-, \mathcal{C})$*-acyclic, for any cotorsion module* C .
(c) $Ext^1_R(M, C) = 0$ *for every* $C \in \mathcal{C} \cap \mathcal{PGF}_{\mathcal{B}}^{\perp}$.
Proof of (b) $\Rightarrow$ (c) Consider a short exact sequence as in (b).

$$0 \to F \to L \to M \to 0$$

Let $C \in \mathcal{C} \cap (\mathcal{PGF}_{\mathcal{B}})^{\perp}$. We have an exact sequence

$$Hom_R(L, C) \xrightarrow{\varphi} Hom_R(F, C) \to Ext^1_R(M, C) \to Ext^1_R(L, C)$$

where $Ext^1_R(L, C) = 0$ since $L \in \mathcal{PGF}_{\mathcal{B}}$, and φ is epic. Hence, $Ext^1_R(M, C) = 0$.

(c) $Ext^1_R(M, C) = 0$ *for every* $C \in \mathcal{C} \cap \mathcal{PGF}_{\mathcal{B}}^{\perp}$.
(d) *There is a short exact sequence of modules*

$$0 \to M \to F \to L \to 0$$

with $F \in \mathcal{F}lat$ *and* $L \in \mathcal{PGF}_{\mathcal{B}}$.

(c) $\Rightarrow$ (d):Consider a short exact sequence

$$0 \to M \to U \to T \to 0$$

with $U \in \mathcal{PGF}_{\mathcal{B}}^{\perp}$ and $T \in \mathcal{PGF}_{\mathcal{B}}$. Let $C \in \mathcal{PGF}_{\mathcal{B}}^{\perp}$ be a cotorsion module. Then, we have an exact sequence

$$Ext_R^1(T, C) \to Ext_R^1(U, C) \to Ext_R^1(M, C)$$

where $Ext_R^1(T, C) = 0$ and $Ext_R^1(M, C) = 0$. Then, $U \in {}^{\perp}(\mathcal{C} \cap \mathcal{PGF}_{\mathcal{B}}^{\perp})$.
Then U has a pure special $\mathcal{PGF}_{\mathcal{B}}$-precover.
- pure exact sequence

$$0 \to K \to L \to U \to 0$$

with $K \in \mathcal{PGF}_{\mathcal{B}}^{\perp}$ and $L \in \mathcal{PGF}_{\mathcal{B}}$.

Then, $L \in \mathcal{PGF}_{\mathcal{B}} \cap (\mathcal{PGF}_{\mathcal{B}})^{\perp}$, so L is projective.
Then U is a pure epimorphic image of a projective module, so $U \in \mathcal{F}lat$. (d) $\Rightarrow$ (a): Follows from the Lemma above.

Corollary

If $\mathcal{B}$ is semi-definable then $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions.

Examples:
1. The class of Gorenstein flat modules is the left half of a complete hereditary cotorsion pair.
2. Consider the class $\mathcal{GF}_{\mathcal{AC}}$ of Gorenstein AC-flat modules. The class $\mathcal{AC}$ of absolutely clean right R-modules is semi-definable. Hence, we have the following properties for Gorenstein AC-flat modules:
- $(\mathcal{GF}_{\mathcal{AC}}, (\mathcal{GF}_{\mathcal{AC}})^{\perp})$ is a complete hereditary cotorsion pair.
- Every module has a Gorenstein AC-flat cover.

Example 3. Consider the class $\mathcal{FI}_n($ of FP_n*-injective* right R-modules defined by Bravo-Perez. Recall that this class is the right orthogonal complement of that of the right R-modules *of type* FP_n, that is, those N for which there is an exact sequence

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$$

where P_k is finitely generated and projective for every $0 \leq k \leq n$.
By Bravo-Perez, $\mathcal{FI}_n$ is a definable class if $n > 1$. Thus, if $\mathcal{GF}_{\mathcal{FI}_n}$ denotes the class of Gorenstein $\mathcal{FI}_n$-flat modules, we have that $\mathcal{GF}_{\mathcal{FI}_n}$ is closed under extensions. As a consequence of the previous results, we have that $(\mathcal{GF}_{\mathcal{FI}_n}, (\mathcal{GF}_{\mathcal{FI}_n})^{\perp})$ is a complete hereditary cotorsion pair.

All the results in this section are joint with S. Estrada and M. Perez

The Gorenstein $\mathcal{B}$-flat stable model category

Given two complete and hereditary cotorsion pairs $(\mathcal{Q}, \mathcal{R}')$ and $(\mathcal{Q}', \mathcal{R})$ in an abelian category $\mathcal{C}$ such that $\mathcal{Q}' \subseteq \mathcal{Q}$, $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{Q}' \cap \mathcal{R} = \mathcal{Q} \cap \mathcal{R}'$, then there exists a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple in $\mathcal{C}$, that is:

(1) $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in $\mathcal{C}$.

(2) $\mathcal{W}$ is *thick*: it is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms between its objects.

By Hovey's correspondence, the existence of such a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ implies the existence of a unique abelian model structure on $\mathcal{C}$ such that:

(1) $\mathcal{Q}$ is the class of cofibrant objects.

(2) $\mathcal{R}$ is the class of fibrant objects.

Let $\mathcal{B}$ be a class of modules that contains the injective right R-modules. We show it is possible to apply the previous result in the setting where:
$\mathcal{Q} := \mathcal{GF}_{\mathcal{B}}(R)$,
$\mathcal{Q}' := \mathcal{F}lat$ the class of flat left R-modules,
$\mathcal{R} := \mathcal{C} = (\mathcal{F}lat)^{\perp}$ the class of cotorsion left R-modules,
$\mathcal{R}' := \mathcal{GC}_{\mathcal{B}}$,

provided that $\mathcal{GF}_{\mathcal{B}}(R)$ is closed under extensions (for instance if $\mathcal{B}$ is a semi-definable class).

Proposition (compatibility between the flat and Gorenstein $\mathcal{B}$-flat cotorsion pairs)

If $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions and $\mathcal{B}$ contains all injective right R-modules, then

$$\mathcal{F}lat \cap \mathcal{C} = \mathcal{GF}_{\mathcal{B}} \cap \mathcal{GC}_{\mathcal{B}}$$

Proof. $(\supseteq)$. Let $M \in \mathcal{GF}_{\mathcal{B}} \cap \mathcal{GC}_{\mathcal{B}}$. Then $M \in \mathcal{C}$. Since M is Gorenstein $\mathcal{B}$-flat, we have a short exact sequence

$$0 \to M \to F \to M' \to 0$$

with F flat, M' is Gorenstein $\mathcal{B}$-flat. This sequence splits, since M is Gorenstein $\mathcal{B}$-cotorsion, so $Ext^1(M', M) = 0$. Hence, M is a direct summand of F, so $M \in \mathcal{F}lat$.

($\subseteq$). Let $N \in \mathcal{F}lat \cap \mathcal{C}$. Then $N \in \mathcal{GF}_{\mathcal{B}}$. Since $(\mathcal{GF}_{\mathcal{B}}, \mathcal{GC}_{\mathcal{B}})$ is complete, there is a short exact sequence

$$0 \to N \to C \to F \to 0$$

with $C \in \mathcal{GC}_{\mathcal{B}}$ and $F \in \mathcal{GF}_{\mathcal{B}}$. Since N and F are Gorenstein $\mathcal{B}$-flat and $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions, we have that $C \in \mathcal{GF}_{\mathcal{B}} \cap \mathcal{GC}_{\mathcal{B}} \subseteq \mathcal{F} \cap \mathcal{C}$. It follows that F is a Gorenstein flat module with finite flat dimension, and so F is flat. Then $Ext^1(F, N) = 0$ since N is cotorsion, and so the previous exact sequence splits. It follows that N is a direct summand of $C \in \mathcal{GC}_{\mathcal{B}}$, and hence $N \in \mathcal{GC}_{\mathcal{B}}$.

Thus we have:

Theorem (the Gorenstein $\mathcal{B}$-flat model structure in $Mod(R)$)

Assume $\mathcal{GF}_{\mathcal{B}}$ is closed under extensions and $\mathcal{B}$ contains all injective right R-modules. Then, there exists a unique abelian model structure on $Mod(R)$ such that $\mathcal{GF}_{\mathcal{B}}$ is the class of cofibrant objects

Corollary (the Gorenstein flat model structure over arbitrary rings)

Over any ring R there exists a unique abelian model structure on $Mod(R)$ such that $\mathcal{GF}$ is the class of cofibrant objects

Corollary

If $\mathcal{B}$ is a semi-definable class of right R-modules that contains the injectives, then, there exists a unique abelian model structure on $Mod(R)$ such that $\mathcal{GF}_{\mathcal{B}}$ is the class of cofibrant objects, $\mathcal{C}$ is the class of fibrant objects, and $\mathcal{PGF}_{\mathcal{B}}^{\perp}$ is the class of trivial objects.

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