Big Projective modules and their applications

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The *stable* category

Let $\mathcal{C}$ be a full subcategory of modules closed by finite direct sums. Fix an object X in $\mathcal{C}$. For any pair of modules M, N in $\mathcal{C}$ consider the following subgroup of $\mathrm{Hom}_R(M,N)$:

$$\mathcal{J}_X(M,N) = \{f\colon M \to N \mid f \text{ factors through } X^{(\Lambda)} \text{ for some set } \Lambda\}$$

$\mathcal{J}_X$ is an ideal of $\mathcal{C}$ and we can consider the quotient category $\mathcal{C}/\mathcal{J}_X$ which has the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{J}_X}(M,N) \colon= \mathrm{Hom}_{\mathcal{C}}(M,N)/\mathcal{J}_X(M,N)$$

Two objects M and N are isomorphic in this quotient category if and only if there exist a set Λ, P and Q in $\mathrm{Add}_\Lambda(X)$ such that $P \oplus M \cong Q \oplus N$

If M, N and X are countably generated, then M and N are isomorphic in the quotient category if and only if $X^{(\omega)} \oplus M \cong X^{(\omega)} \oplus N$.

We extend Dress equivalence to the *stable* category

Theorem

Let M be a finitely generated right module over ANY ring R, and let $S = \mathrm{End}_R(M)$. Let X be an object of $\mathrm{Add}(M)$. Let $P_S = \mathrm{Hom}_R(M, X)$, and let $I = \mathrm{Tr}_S(P) = \sum_{f \in \mathrm{Hom}_S(P,S)} f(P)$.
Then, the functor $\mathrm{Hom}_R(M, -) \otimes_S S/I$ induces an equivalence between the categories $\mathrm{Add}(M)/\mathcal{J}_X$ and $\mathrm{Add}(S/I)$.
The equivalence restricts well to countably generated objects.

The key result: Projective modules can be lifted modulo the trace ideal of a projective module.

We emphasize on $\mathrm{add}\,(S/I)$: If X is countably generated, a finitely generated projective right S/I-module Q corresponds via the equivalence to a countably generated object Y of the stable category. We lift it to $\mathrm{Add}_{\aleph_0}(M)$ as $Y \oplus X^{(\omega)}$. **Such lifting is unique up to isomorphism because it comes from a uniquely determined I-big projective S-module.**

The case of finitely generated noetherian algebras

Recall: All countably generated projective S-modules are relatively big in this context and they are determined by a suitable idempotent ideal I (depending on the projective) and a finitely generated projective module of S/I

Recipe to compute the objects of $\mathrm{Add}(M)$ for M finitely generated

Remark: Enough to compute countably generated objects.

1. Compute $S = \mathrm{End}_R(M)$. When R is a finitely generated noetherian algebra, so is S.
2. Compute the idempotent ideals of S. The set idempotent ideals coincides with the set of trace ideals of projective modules.
3. For any idempotent ideal I of S, compute the finitely generated projective modules over S/I.
4. **"Glue" everything together.**

We get all the information on infinitely generated modules just out of finitely generated data!!

Thanks for your attention!!