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Reconstructing projective modules from its trace ideal



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ABSTRACT

We make a detailed study of idempotent ideals that are traces of countably generated projective right modules. We associate to such ideals an ascending chain of finitely generated left ideals and, dually, a descending chain of cofinitely generated right ideals.

The study of the first sequence allows us to characterize trace ideals of projective modules and to show that projective modules can always be lifted modulo the trace ideal of a projective module. As a consequence we give some new classification results of (countably generated) projective modules over particular classes of semilocal rings. The study of the second sequence leads us to consider projective modules over noetherian FCR-algebras; we make some constructions of non-trivial projective modules showing that over such rings the behavior of countably generated projective modules that are not direct sum of finitely generated ones is, in general, quite complex.

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It was proved by Whitehead [22] that an idempotent ideal that is finitely generated as a left ideal is the trace of a countably generated projective right R -module. For example, if R is a left noetherian ring then any idempotent ideal is the trace ideal of a countably generated projective right R -module. Příhoda in [13] developed a machinery to work with countably generated projective modules in the setting of noetherian rings. These tools have been extremely helpful in describing non-finitely generated projective modules over suitable classes of noetherian rings with low Krull dimension and not too many idempotent ideals. See [13] for applications to integral group algebras and to some enveloping algebras of Lie algebras, also in [16,15] generalized Weyl algebras and some examples of lattices are studied. Finally, we mention [8] where the behavior of projective modules over semilocal noetherian rings is completely described.

One of the main ideas in Příhoda's techniques is that projective modules over suitable rings can be reconstructed by determining trace ideals and the finitely generated projective modules modulo such trace ideals. In this paper we extend these results on trace ideals of projective modules to a non-necessarily noetherian setting. However, the not so exciting news are that certainly further ideas will be needed to be able to have such complete classifications of projective modules also for some classes of non-noetherian rings. For example, just to understand completely projective modules over semilocal rings seems to be quite a hard problem.

To explain which are the particular properties of trace ideals of projective modules, let us make some observations on trace ideals of the finitely generated ones. Let P be a finitely generated projective right module over a ring R , and let E be an idempotent $n \times n$ matrix with entries in R such that $P \cong ER^n$. The left ideal J generated by the entries of E is finitely generated and satisfies that $J^2 = J$ and the same happens if we consider K to be the right ideal of R generated by the entries of E . Moreover, if I is the trace of P then $I = JR = RK$.

Let I denote now the trace ideal of a countably generated projective right R -module P . Let E be a column-finite countable idempotent matrix defining P , then considering the left ideals generated by the entries in the first columns of E one constructs an ascending chain of finitely generated left ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n \geq 1} J_n R = I$. The existence of such chains characterizes the ideals that are traces of countably generated projective right R -modules, cf. Proposition 2.4. In Proposition 2.6, we show that trace ideals of arbitrary projective right R -modules can be characterized in terms of the existence of a direct system of ideals having such ascending sequences of finitely generated left ideals. As a consequence, we prove in Corollary 3.2 that projective modules always lift modulo trace ideals of projective module.

Further applications are developed in Section 4 where we describe countably generated projective modules over a particular class of semilocal rings that seems to be specially close to the class of semiperfect rings: the class of semilocal rings such that for any simple right R -module V there exists $n_V \geq 1$ such that V^{n_V} has a projective cover. We call such a ring semi-semiperfect. We give examples showing that, even being close to be semiperfect, semi-semiperfect rings can have countably generated projective modules that are not a direct sum of finitely generated ones; cf. [Example 4.7](#).

Going back again to a column-finite countable idempotent matrix E defining a countably generated projective right R -module P , we consider the right ideals generated by the entries in all rows of E , except perhaps a finite number of them. With these ideals it is possible to construct a descending chain of right ideals $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ such that $K_{n+1}K_n = K_{n+1}$ and P/PK_n is finitely generated. The existence and the properties of this descending chain gives us a curious phenomena: if there exists an ideal K of R minimal with respect to the property P/PK is finitely generated then there exists n such that $K = RK_n$ and, hence, K is an idempotent ideal. When this happens P is said to be a fair sized projective module [[13](#)]. These ideas and some consequences are developed in Section 5. Among other things we prove that over R a semilocal ring any projective module P is fair sized ([Proposition 5.6](#)), but we fail to understand the rôle of the idempotent ideal K associated to P .

If R is a noetherian ring (semilocal or not) and P is a fair sized projective module then the idempotent ideal K is also the trace of a projective module. This can be quite helpful in determining the structure of P . For example, if R is a noetherian ring such that all countably generated projective right modules that are not direct sums of finitely generated ones are fair sized, and there is only a finite number of idempotent ideals then, as it was shown in [[13](#)], the projective modules over such rings are determined by pairs (\bar{P}, I) where I is an idempotent ideal (hence the trace of a projective module) and \bar{P} is a finitely generated R/I -projective module.

However, over a noetherian ring not all countably generated projective modules that are not direct sums of finitely generated ones are fair sized. In Sections 6 and 7 we review the construction of a projective module with prescribed trace ideal due to Whitehead [[22](#)] and we prove that if R has an infinite descending chain of idempotent ideals with semisimple factors and starting at I , then R has uncountably many non-isomorphic, non-fair sized projective modules with trace ideal I . Nice examples where this result can be applied are the enveloping algebras of finite dimensional semisimple Lie algebras over an algebraic closed field and also their quantum deformations. More generally, noetherian FCR -algebras are the right context for these applications. As recalled in [Lemma 7.2](#), any nonzero finitely generated projective module over the enveloping algebra of a finite dimensional semisimple Lie algebra is a generator and, hence, its trace is the whole ring. Therefore the projective modules in our construction are not direct sums of finitely generated ones, so it looks really challenging to try to describe all projective modules over such rings.

1. On Whitehead’s characterization of countably generated projective modules

All our rings are associative with 1, and ring morphism means unital ring morphism.

We think on (countably generated) projective modules as (countable) direct limits of finitely generated projective modules (i.e. as flat modules) that are projective. To characterize when this happens it is important to keep in mind the following well known result. A complete proof of it can be found in [1, Proposition 2.1].

Lemma 1.1. *Let R be a ring. Let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots M_n \xrightarrow{f_n} M_{n+1} \dots$ be a countable direct system of right R -modules such that $M = \varinjlim M_n$. Then the canonical presentation of the direct limit $0 \rightarrow \bigoplus_{n \geq 1} M_n \rightarrow \bigoplus_{n \geq 1} M_n \rightarrow M \rightarrow 0$ splits if and only if there exists a sequence of homomorphisms*

$$\dots M_{n+1} \xrightarrow{g_n} M_n \rightarrow \dots \rightarrow M_2 \xrightarrow{g_1} M_1$$

such that:

For any $n \geq 1$ and for any finite subset $Y \subseteq M_n$ there exists $\ell = \ell(n, Y) > n$ such that, for any $m > \ell$ and any $y \in Y$, $g_m f_m \dots f_n(y) = f_{m-1} \dots f_n(y)$

Our construction of countably generated projective modules relies on Proposition 1.4 which is due to Whitehead [22, Theorem 1.9]. We want to explain how this result can be seen as a consequence of the characterization of countably generated projective modules as countably generated flat Mittag–Leffler modules [18]. To clarify the relation between the two approaches we formulate the following lemma.

Lemma 1.2. *(See Raynaud, Gruson [18].) Let R be a ring. Let $R^{m_1} \xrightarrow{f_1} R^{m_2} \xrightarrow{f_2} \dots R^{m_k} \xrightarrow{f_k} R^{m_{k+1}} \dots$ be a countable direct system of finitely generated free right modules with limit P . Then the following statements are equivalent:*

- (i) For any $k \geq 1$ the descending chain of abelian groups

$$\begin{aligned} \text{Hom}_R(R^{m_{k+1}}, R) f_k &\supseteq \text{Hom}_R(R^{m_{k+2}}, R) f_{k+1} f_k \supseteq \dots \\ &\supseteq \text{Hom}_R(R^{m_{k+\ell+1}}, R) f_{k+\ell} \dots f_k \supseteq \dots \end{aligned}$$

is stationary.

- (ii) There exists a sequence of natural numbers $(\ell_k)_{k \geq 1}$ and a sequence of module homomorphisms $(g_k: R^{m_{k+\ell_k+1}} \rightarrow R^{m_{k+\ell_k}})_{k \geq 1}$ such that, for any $k \geq 1$ and any $n \geq k + \ell_k$, $g_n f_n \dots f_k = f_{n-1} \dots f_k$.
- (iii) For any right module M , and for any $k \geq 1$ the descending chain of abelian groups

$$\begin{aligned} \text{Hom}_R(R^{m_{k+1}}, M) f_k &\supseteq \text{Hom}_R(R^{m_{k+2}}, M) f_{k+1} f_k \supseteq \dots \\ &\supseteq \text{Hom}_R(R^{m_{k+\ell+1}}, M) f_{k+\ell} \dots f_k \supseteq \dots \end{aligned}$$

is stationary.

- (iv) P is projective.

Moreover, when the above equivalent statements hold, any direct system of finitely generated projective modules with limit P and of the form

$$F_1 \xrightarrow{f'_1} F_2 \xrightarrow{f'_2} \dots F_k \xrightarrow{f'_k} F_{k+1} \dots$$

satisfies conditions (i), (ii) and (iii).

Proof. Assume (i). Fix $k \geq 1$, and set $\ell_k \geq 1$ such that $\text{Hom}_R(R^{m_k+\ell_k+1}, R)f_{k+\ell_k} \cdots f_k = \text{Hom}_R(R^{m_k+\ell_k}, R)f_{k+\ell_k-1} \cdots f_k$.

For any $i \in \{1, \dots, m_k+\ell_k\}$, let $\pi_i: R^{m_k+\ell_k} \rightarrow R$ denote the projection onto the i -th component, and let $\varepsilon_i: R \rightarrow R^{m_k+\ell_k}$ denote the canonical inclusion defined by $\varepsilon_i(r) = (0, \dots, r^i, \dots, 0)$ for any $r \in R$. By hypothesis, there exists $\omega_i \in \text{Hom}_R(R^{m_k+\ell_k+1}, R)$ such that

$$\pi_i f_{k+\ell_k-1} \cdots f_k = \omega_i f_{k+\ell_k} \cdots f_k.$$

Now $g_k = \sum_{i=1}^{m_k+\ell_k} \varepsilon_i \omega_i$ satisfies the desired properties.

It is clear that (ii) implies (iii), and that (iii) implies (i). Using Proposition 1.1 it is easy to see that (ii) holds if and only if P is projective.

The final part of the statement follows in a similar way [18]. \square

Lemma 1.3. *Let R be a ring. Let $R^{m_1} \xrightarrow{f_1} R^{m_2} \xrightarrow{f_2} \dots R^{m_k} \xrightarrow{f_k} R^{m_{k+1}} \dots$ be a countable direct system of finitely generated free right modules with limit P . For any $k \geq 1$, let $u_k: R^{m_k} \rightarrow P$ denote the canonical map. Then the following statements are equivalent:*

- (i) *For any $k > 1$ there exist a module homomorphism $g_k: R^{m_{k+1}} \rightarrow R^{m_k}$ such that $g_k f_k f_{k-1} = f_{k-1}$.*
- (ii) *For any $k \geq 1$, $\text{Hom}_R(P, R)u_k = \text{Hom}_R(R^{m_{k+1}}, R)f_k$.*

When the above statements hold

$$\text{Tr}(P) = \sum_{k \geq 1} \left(\sum_{i=1}^{m_{k+1}} R \pi_i f_k (R^{m_k}) \right)$$

where $\pi_i: R^{m_{k+1}} \rightarrow R$ denotes the projection onto the i -th component. In particular if, for any $k \geq 1$, f_k is given by left multiplication by the matrix $X_k = (x_{ij}^k)$ then $\text{Tr}(P) = \sum_{k,i,j} R x_{ij}^k R$.

Proof. (i) \Rightarrow (ii). We only need to prove that for any $k \geq 1$,

$$\text{Hom}_R(P, R)u_k \supseteq \text{Hom}_R(R^{m_{k+1}}, R)f_k$$

Fix $k \geq 1$ and $\omega \in \text{Hom}_R(R^{m_{k+1}}, R)$. By (i), it follows that the sequence of homomorphisms $\{\omega f_k, \omega g_{k+1} f_{k+1}, \omega g_{k+1} g_{k+2} f_{k+2}, \dots\}$ induces a homomorphism $f: \varinjlim R^{m_{k+\ell}} = P \rightarrow R$ such that $f u_k = \omega f_k$.

(ii) \Rightarrow (i). Since for any $k \geq 1$, $\text{Hom}_R(P, R) u_k \subseteq \text{Hom}_R(R^{m_{k+2}}, R) f_{k+1} f_k$, it follows from (ii) that $\text{Hom}_R(R^{m_{k+1}}, R) f_k = \text{Hom}_R(R^{m_{k+2}}, R) f_{k+1} f_k$. Therefore Lemma 1.2(i) holds and, for any $k \geq 1$, the ℓ_k of Lemma 1.2(ii) is equal to 1. Hence (i) holds.

It remains to prove the statement on the trace of P . It is clear that $\sum_{i=1}^{m_{k+1}} R \pi_i f_k(R^{m_k}) \subseteq \text{Tr}(P)$. Let $f \in \text{Hom}_R(P, R)$. For any $p \in P$ there exists $k \geq 1$ and $x \in R^{m_k}$ such that $p = u_k(x)$. Hence, by (ii), there exists $\omega \in \text{Hom}_R(R^{m_{k+1}}, R)$ such that

$$f(p) = f \circ u_k(x) = \omega \circ f_k(x) \in \sum_{i=1}^{m_{k+1}} R \pi_i f_k(R^{m_k}). \quad \square$$

Now we sketch the proof of Whitehead’s characterization.

Proposition 1.4. (See [22, Theorem 1.9].) *Let R be a ring, and let P_R be a countably presented (or generated) flat right R -module. Then the following statements are equivalent:*

- (i) P_R is projective.
- (ii) There exists a direct system of finitely generated free modules

$$R^{m_1} \xrightarrow{X_1} R^{m_2} \xrightarrow{X_2} \dots R^{m_k} \xrightarrow{X_k} R^{m_{k+1}} \dots$$

with limit P , and where $X_k: R^{m_k} \rightarrow R^{m_{k+1}}$ denotes the homomorphism given by left multiplication by the matrix X_k , and a sequence of matrices $\{Y_k\}_{k>1}$ such that $Y_k \in M_{m_k \times m_{k+1}}(R)$ and satisfies that $Y_k X_k X_{k-1} = X_{k-1}$ for any $k > 1$.

In this situation if, for any $k \geq 1$, $X_k = (x_{ij}^k)$ then $\text{Tr}(P) = \sum_{k,i,j} R x_{ij}^k R$.

Proof. Assume (i). Write P as the direct limit of

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots F_k \xrightarrow{f_k} F_{k+1} \dots$$

where F_k is finitely generated and free for any $k \geq 1$. By Lemma 1.2, there exist a sequence of natural numbers $(\ell_k)_{k \geq 1}$ and a sequence of module homomorphisms $(g_k: F_{k+\ell_k+1} \rightarrow F_{k+\ell_k})_{k \geq 1}$ such that, for any $k \geq 1$ and any $n \geq k + \ell_k$, $g_n f_n \dots f_k = f_{n-1} \dots f_k$. As done by Whitehead in [22, Theorem 1.9] (or also as in [2]), this allows to rearrange the direct system above to one with the same limit and with the properties claimed in (ii).

Lemma 1.2, also shows that (ii) implies (i).

The rest of the statement follows from Lemma 1.3. \square

2. An ascending chain of finitely generated left ideals

Next lemma explains how to construct an ascending chain of finitely generated left ideals out of a column finite idempotent matrix of countable size.

Lemma 2.1. *Let R be a ring, and let I be a two-sided ideal of R which is the trace ideal of a countably generated projective right R -module. Then there exists an ascending chain of finitely generated left ideals $J_1 \subseteq J_2 \subseteq \dots \subseteq J_n \subseteq \dots$ satisfying that, for any $n \geq 1$, $J_{n+1}J_n = J_n$ and such that $I = \bigcup_{n \geq 1} J_n R$.*

Proof. Let P be the countably generated projective right module such that $I = \text{Tr}(P)$. Let $A = (a_{ij})_{i,j \geq 1}$ be a column finite countable idempotent matrix with entries in R such that $P = AR^{(\mathbb{N})}$. Note that $I = \sum_{i,j \geq 1} Ra_{ij}R$.

For each $n \geq 1$ consider the finitely generated left ideal $L_n = \sum_{j \leq n} Ra_{ij}$. Since A is a column finite idempotent matrix, for each $n \geq 1$, there exists $k_n > n$ such that $L_{k_n}L_n = L_n$. Therefore by choosing a suitable subchain of $\{L_n\}_{n \geq 1}$ we can construct an ascending chain of finitely generated left ideals $\{J_n\}_{n \geq 1}$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n \geq 1} J_n R$. This finishes the proof of the lemma. \square

Reformulating a result by Whitehead [22, Theorem 2.5], we shall see that the existence of an ascending chain as in Lemma 2.1 characterizes the ideals that are traces of countably generated projective right R -modules.

It is useful to keep in mind the following lemma, as it explains some modifications that can be made in the ascending chains appearing in Lemma 2.1.

Lemma 2.2. *Let R be a ring. Let $J_1 \subseteq J_2$ be finitely generated left ideals of R satisfying that $J_2J_1 = J_1$. For $i = 1, 2$, fix A_i a finite set of generators of J_i .*

(i) *Let X be a finite subset of R such that $1 \in X$. For $i = 1, 2$, set*

$$J'_i = \sum_{r \in X} \sum_{a \in A_i} Rar.$$

Then $J'_1 \subseteq J'_2$ and $J'_2J'_1 = J'_1$. Moreover, for $i = 1, 2$, $J_i \subseteq J_i R = J'_i R$.

(ii) *There exists a finite $X \subseteq R$ with $1 \in X$ such that J_1 is generated by $B \cdot A_1 = \{b \cdot a \mid a \in A_1 \text{ and } b \in B\}$ where*

$$B = A_2 \cdot X = \{ar \mid a \in A_2, r \in X\}.$$

Moreover, if we set $J'_2 = \sum_{b \in B} Rb$ then $J'_2J_1 = J_1$ and $J_2R = J'_2R$.

Proof. Statement (i) is immediate. To prove (ii) observe that for any $a \in A_1$

$$a = \sum_{a' \in A_1} y_{a'}^a a'$$

where $y_{a'}^a \in J_2 R$ so that $y_{a'}^a$ is a sum of elements of the form sbr with $b \in A_2$ and $r, s \in R$. Let X be a finite set of R that contains 1 and all the elements r in the expressions of the elements $y_{a'}^a$. Then X satisfies the claimed properties. \square

Lemma 2.3. *Let R be a ring. Let J_1 and J_2 be left ideals of R such that there exist $a_1, \dots, a_\ell \in J_1$ satisfying that $J_1 = \sum_{i=1}^\ell J_2 a_i$. Set $a = \begin{pmatrix} a_1 \\ \vdots \\ a_\ell \end{pmatrix}$. Fix $k \geq 1$ and let*

$$A = \begin{pmatrix} a & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & a \end{pmatrix} \in M_{k \cdot \ell \times k}(J_1).$$

Then, for any $m \geq 1$, $M_{m \times k}(J_1) = M_{m \times k \cdot \ell}(J_2)A$.

Proof. Clearly, $M_{m \times k \cdot \ell}(J_2)A \subseteq M_{m \times k}(J_1)$. To prove the reverse inclusion let $B = (b_{ij}) \in M_{m \times k}(J_1)$. We may assume that B has only one nonzero entry, $b_{i_0 j_0}$ say.

As $J_1 = \sum_{i=1}^\ell J_2 a_i$, $b_{i_0 j_0} = \sum_{i=1}^\ell b_i a_i$ with $b_i \in J_2$ for $i = 1, \dots, \ell$. Denote the entries of A by a_{ij} then, as any column of A contains a , there exists $i_1 \geq 0$ such that

$$a = \begin{pmatrix} a_{i_1+1 j_0} \\ \vdots \\ a_{i_1+\ell j_0} \end{pmatrix}$$

and all other entries of the j_0 column of A are zero. Let $C = (c_{ij}) \in M_{m \times k \cdot \ell}(J_2)$ be the matrix such that all its entries are zero except possibly for $c_{i_0 i_1+1} = b_1, \dots, c_{i_0 i_1+\ell} = b_\ell$. Then $C \cdot A = B$, which shows that $B \in M_{m \times k \cdot \ell}(J_2)A$. \square

Proposition 2.4. *Let R be a ring, and let I be a two-sided ideal of R . Then the following statements are equivalent:*

- (i) I is the trace ideal of a countably generated projective right R -module.
- (ii) There exists an ascending chain of finitely generated left ideals $\{J_n\}_{n \geq 1}$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n \geq 1} J_n R$.
- (iii) There exists an ascending chain of finitely generated left ideals $\{J'_n\}_{n \geq 1}$ such that $I = \bigcup_{n \geq 1} J'_n R$, together with a suitable choice of a finite set of generators B_n of J'_n such that $B_{n+1} \cdot B_n = \{b \cdot b' \mid b \in B_{n+1} \text{ and } b' \in B_n\}$ generates J'_n . In particular, $J'_{n+1}J'_n = J'_n$.

Proof. (i) \Rightarrow (ii) and (iii) \Rightarrow (i) follow from [22, Theorem 2.5]. Lemma 2.1 gives a different approach to (i) \Rightarrow (ii). We also include a proof of (iii) \Rightarrow (i).

For every $n \geq 1$, set $Y_n = \begin{pmatrix} b_1^n \\ \vdots \\ b_n^n \end{pmatrix}$. Define $X_1 = Y_1$, and for $n > 1$

$$X_n = \begin{pmatrix} Y_n & \dots & 0 \\ \vdots & \ddots \ell_{n-1} \cdots \ell_1 & \vdots \\ 0 & \dots & Y_n \end{pmatrix}.$$

By Lemma 2.3, for any $n \geq 1$

$$X_n \in M_{(\ell_n \cdots \ell_1) \times (\ell_n \cdots \ell_1)}(J'_{n+1})X_n = M_{(\ell_n \cdots \ell_1) \times (\ell_{n+1} \cdots \ell_1)}(R)X_{n+1}X_n.$$

So that, for any $n \geq 1$, there exists a matrix C_n such that $C_n X_{n+1} X_n = X_n$.

By Proposition 1.4, the limit of the direct system

$$R \xrightarrow{X_1} R^{\ell_1} \xrightarrow{X_2} R^{\ell_2 \ell_1} \dots$$

is a countably generated projective module with trace I .

(ii) \Rightarrow (iii). Let $\{J_n\}_{n \geq 1}$ be an ascending chain of finitely generated left ideals with the properties claimed in (ii). Then $\{J'_n\}_{n \geq 1}$ and $\{B_n\}$ are constructed by suitably enlarging the left ideals J_n by repeatedly applying Lemma 2.2. \square

Our aim now is to give a characterization of trace ideals of arbitrary projective modules. Next lemma gives a more intrinsic approach to the construction of the sequence of Lemma 2.1.

Lemma 2.5. *Let R be a ring. Let P be a projective right R -module and let $I = \text{Tr}(P)$. Then, for any finite subset X of I there exist finitely generated left ideals $J_1 \leq J_2 \leq I$ such that $X \subseteq J_1$ and $J_2 J_1 = J_1$.*

Proof. Since X is finite, it is contained in $\text{Tr}(P')$ where P' is a countably generated direct summand of P . Thus, without loss of generality, we may assume that P is countably generated.

Fix $(p_i, \omega_i)_{i \geq 1}$ to be a dual basis of P . That is for any $i \geq 1$, $p_i \in P$ and $\omega_i \in \text{Hom}_R(P, R)$, and for any $p \in P$, $p = \sum_{i \geq 1} p_i \omega_i(p)$ where $\omega_i(p) = 0$ for almost all $i \geq 1$. Notice that, for any $f \in \text{Hom}_R(P, R)$ and any $p \in P$, $f(p) = \sum_{i \geq 1} f(p_i) \omega_i(p)$. That is, there exists $n_p \geq 1$ such that the left ideal of R

$$J_p = \sum_{f \in \text{Hom}_R(P, R)} f(p) = \sum_{i=1}^{n_p} R \omega_i(p) \leq I$$

is finitely generated and satisfies that $J'_p J_p = J_p$ where $J'_p = \sum_{i=1}^{n_p} J_{p_i}$; hence J'_p is also finitely generated.

Our previous argument shows that there exists a finite set $X_0 \subseteq P$ such that $X \subseteq \sum_{p \in X_0} J_p = J_1$ and that the left ideal J_1 is finitely generated; moreover $J_2 = J_1 + \sum_{p \in X_0} J'_p$ is also finitely generated, and $J_2 J_1 = J_1$. \square

Proposition 2.6. *Let R be a ring, and let I be an ideal of R . Then the following statements are equivalent:*

- (i) *There exists a projective right R -module P such that $I = \text{Tr}(P)$.*
- (ii) *For any finite subset X of I there exists a couple of finitely generated left ideals $J_1 \leq J_2 \leq I$ such that $X \subseteq J_1$ and $J_2 J_1 = J_1$.*
- (iii) *There exists a left ideal $J \leq I$, such that $JR = I$ and for any finite subset X of J there exist finitely generated left ideals $J_1 \leq J_2 \leq J$ such that $X \subseteq J_1$ and $J_2 J_1 = J_1$.*

Moreover, when the above equivalent statement hold, I is an \aleph_1 directed union of ideals of the form $\bigcup_{n \geq 1} J_n R$ where $J_1 \leq J_2 \leq \dots \leq J_n \leq \dots$ is a sequence of finitely generated left ideals such that $J_{n+1} J_n = J_n$ for any $n \geq 1$.

Proof. (i) \Rightarrow (ii) follows from Lemma 2.5. The implication (ii) \Rightarrow (iii) is trivial. Now we prove (iii) \Rightarrow (i).

Let $\Lambda = \{K \leq J \mid K \text{ is finitely generated and there exists } K' \leq J \text{ also finitely generated such that } K'K = K\}$. Let \mathcal{C} be the set of ascending chains in Λ

$$J_1 \subseteq J_2 \subseteq \dots \subseteq J_n \subseteq \dots$$

satisfying that $J_{n+1} J_n = J_n$ for any $n \geq 1$. By Proposition 2.4, for any $C \in \mathcal{C}$ there exists a countably generated projective right R -module P_C such that $\text{Tr}(P_C) = \sum_{J_n \in C} J_n R$. Set $P = \sum_{C \in \mathcal{C}} P_C$, we claim that $I = \text{Tr}(P)$. It is clear that $I \supseteq \text{Tr}(P)$, to see the opposite inclusion, notice that, Lemma 2.5, for any finite subset X of J there exists $C \in \mathcal{C}$ such that $X \subseteq \sum_{J_n \in C} J_n$ and, hence $X \subseteq \text{Tr}(P_C)$. This finishes the proof of the statement.

The proof of the remaining part of the statement follows from (iii) \Rightarrow (i). \square

Corollary 2.7. *Let R be a ring. Let J be a finitely generated left ideal such that $J^2 = J$ then JR is the trace of a (countably generated) projective right R -module.*

Proof. The ideal $I = JR$ fulfills Proposition 2.4(ii) since the ascending chain can be taken to be $J_n = J$ for any $n \geq 1$. \square

Corollary 2.8. *Let R be a ring such that $R/J(R)$ satisfies the ascending chain condition on two-sided ideals (e.g. $R/J(R)$ left or right noetherian). Then an ideal I is the trace*

of a (countably generated) projective right R -module if and only if there exists a finitely generated left ideal J such that $J^2 = J$ and $I = JR$.

Proof. Assume that I is the trace ideal of projective right R -module. Let $\{J_n\}_{n \geq 1}$ be a sequence of finitely generated left ideals such that $J_{n+1}J_n = J_n$ for any $n \geq 1$ and $I \supseteq \bigcup_{n \geq 1} J_n R$. We are going to see that such sequence is stationary. Consider the ascending chain of two-sided ideals $\{J_n R\}_{n \geq 1}$. As $R/J(R)$ satisfies the ascending chain condition on two-sided ideals, there exists n_0 such that $J_{n_0} R + J(R) = J_{n_0+k} R + J(R)$ for any $k \geq 0$.

For any $n \geq n_0$,

$$J_n = (J_{n+1} + J(R))J_n = (J_{n+1}R + J(R))J_n = (J_n R + J(R))J_n = J_n^2 + J(R)J_n.$$

As J_n is finitely generated we can apply Nakayama’s Lemma to deduce that $J_n = J_n^2$ for any $n \geq n_0$.

Moreover, for any $n \geq n_0$,

$$J_n = J_n^2 = (J_n R + J(R))J_n = (J_{n_0} R + J(R))J_n = J_{n_0} J_n + J(R)J_n$$

again Nakayama’s Lemma allows us to deduce that $J_n = J_{n_0} J_n$. Therefore, for any $n \geq n_0$, $J_n R = J_{n_0} J_n R = J_{n_0} R$.

Now if I is not of the form JR for a finitely generated left ideal J then [Lemma 2.5](#) would lead to the construction of a strictly ascending chain $\{J_n\}_{n \geq 1}$ of finitely generated left ideals such that $J_{n+1}J_n = J_n$ and $J_n R \neq J_{n+1} R$ which is impossible by the above argument. Therefore there exists a finitely generated left ideal J such that $JR = I$. By [Lemma 2.5](#), there exists a finitely generated left ideals $J_1 \leq J_2 \leq I$ such that $J \leq J_1$ and $J_2 J_1 = J_1$. Again by [Lemma 2.5](#), there exists finitely generated ideals $J_2' \leq J_2 \leq J_3 \leq I$ such that $J_3 J_2 = J_2$. This way we construct an ascending chain $J_1 \leq J_2 \leq J_3 \leq \dots$ which, by the previous argument, it is stationary at a finitely generated ideal J_{n_0} such that $J_{n_0}^2 = J_{n_0}$ and, since $J_1 \leq J_{n_0}$, also $I = J_{n_0} R$. \square

Corollary 2.9. *Let R be a ring, and let I and I' be ideals of R . Assume that I is the trace of a projective right R -module. If $I + J(R) \subseteq I' + J(R)$ then $I \subseteq I'$.*

In particular, if I' is also the trace ideal of a projective right R module then $I + J(R) = I' + J(R)$ if and only if $I = I'$.

Proof. Choose an ascending chain of finitely generated left ideals $\{J_n\}_{n \geq 1}$ satisfying that, for any n , $J_{n+1}J_n = J_n$ and such that $I \supseteq \bigcup_{n \geq 1} J_n R$.

Fix $n \geq 1$, then

$$J_n = (J_{n+1} + J(R))J_n = (I' + J(R))J_n = I' J_n + J(R)J_n,$$

by Nakayama's Lemma, $J_n = I'J_n \subseteq I'$ for any $n \geq 1$. Hence, $\bigcup_{n \geq 1} J_n R \subseteq I'$. Since by Proposition 2.6, I is a directed union of unions of such sequences, we deduce that $I' \subseteq I$. \square

Remark 2.10. (1). It was proved by Sakhaev [19] that I is a countably generated pure right ideal of R if and only if I can be generated by a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of R satisfying that $a_{n+1}a_n = a_n$ for any $n \in \mathbb{N}$. Since in this situation R/I is flat and countably presented, it follows from a result of Jensen that R/I has projective dimension at most one, so that I_R is projective. In particular, $RI = \text{Tr}(I)$ is the trace of a countably generated projective right R -module. Notice that, for such RI , the sequence of ideals given by Proposition 2.4 can be taken to be $J_n = Ra_n$ for any $n \in \mathbb{N}$.

(2). Let I be a pure right ideal of R . Consider

$$\mathcal{S} = \{L \leq I \mid L \text{ is countably generated and pure in } R\}.$$

In view of (1) (or also because I_R is a flat Mittag-Leffler module), the set \mathcal{S} is directed, closed under countable unions and $I = \sum_{L \in \mathcal{S}} L$. Since any $L \in \mathcal{S}$ is a projective right ideal, we deduce that RI is the trace ideal of the projective module $P = \bigoplus_{L \in \mathcal{S}} L$. So that, for any pure right ideal I the two-sided ideal RI is the trace of a projective right R -module.

(3). In general, trace ideals of (finitely generated) projective modules are neither right nor left pure. For example, let k be a field and let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. The artinian ring R has two idempotents $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The indecomposable projective right R -modules are $P_1 = e_1 R = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$, and the indecomposable projective left R -modules are $Q_1 = R e_1 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ and $Q_2 = R e_2 = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$. Therefore $P_1 = \text{Tr}(P_1) = \text{Tr}(Q_1)$ is pure as a right ideal but not as a left ideal, while $Q_2 = \text{Tr}(P_2) = \text{Tr}(Q_2)$ is pure as a left ideal but not as a right ideal.

Now in the ring $R \times R$ the two-sided ideal generated by the idempotent (e_1, e_2) is a trace ideal of a projective module that is neither right nor left pure.

(4). Not all trace ideals of projective modules are obtained as in (2). For example, Sakhaev constructed a ring R such that $R_R/J(R)$ is a direct sum of 2 non-isomorphic simple modules S_1, S_2 , there is no projective R -module Q such that $Q/QJ(R)$ is simple and there exists a projective module P such that $P/PJ(R) \simeq S_1^2$; see also [9]. We claim that such ring has no non-trivial pure right ideals, and hence $\text{Tr}(Q) \neq RI$ for a pure ideal I of R .

Indeed, if R has a pure right ideal different from R and $\{0\}$ then it has a non-trivial countably generated pure right ideal; call such an ideal I . By (1), I is projective. Since $I/IJ(R)$ is a submodule of $R/J(R)$ and it is neither isomorphic to S_1 nor S_2 , then $I = 0$ or $I = R$, a contradiction.

For a commutative ring R , Vasconcelos showed that trace ideals of projective modules over R are precisely the pure ideals. We give an alternative proof of Vasconcelos result.

Lemma 2.11. *Let R be a commutative ring. Let $J_1 \subseteq J_2$ be ideals such that J_1 is finitely generated and $J_2J_1 = J_1$. Then there exists $a \in J_2$ such that $ab = b$ for any $b \in J_1$.*

Proof. Let b_1, \dots, b_r be a finite set of generators of J_1 . By hypothesis, for any $i = 1, \dots, r$, there exists $a_{ij} \in J_2$ with $i = 1, \dots, r$, such that $b_i = \sum_{j=1}^r a_{ij}b_j$. So that, if $A = (a_{ij}) \in M_r(J_2)$ then

$$(\text{Id}_n - A) \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = 0.$$

Since $\det(\text{Id}_n - A) = 1 - a$ with $a \in J_2$, we deduce that $ab_i = b_i$ for any $i = 1, \dots, r$. So that a has the claimed property. \square

Proposition 2.12. *Let R be a commutative ring, and let I be an ideal of R . The following are equivalent statements:*

- (i) *I is the trace ideal of a countably generated projective module.*
- (ii) *I can be generated by a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of R satisfying that $a_{n+1}a_n = a_n$ for any $n \in \mathbb{N}$.*
- (iii) *I is a countably generated pure ideal of R .*
- (iv) *I is a countably generated, projective and pure ideal of R .*

Proof. In view of Remark 2.10, it is enough to show that (i) \Rightarrow (ii). By Proposition 2.4, $I = \bigcup_{n \in \mathbb{N}} J_n$ where J_n are finitely generated ideals of R satisfying that $J_{n+1}J_n = J_n$. By repeatedly using Lemma 2.11 we find that, for any $n \in \mathbb{N}$, there exists $a_n \in J_{n+1}$ such that $a_nb = b$ for any $b \in J_n$. The sequence $(a_n)_{n \in \mathbb{N}}$ satisfies the desired properties. \square

Corollary 2.13. *Let R be a commutative ring. An ideal I of R is the trace of a projective R -module if and only if it is pure in R .*

Proof. In view of Remark 2.10, it is enough to show that if I is an ideal of R that is the trace of a projective module P then it is pure. Since any projective module is a direct sum of countably generated projective submodules, we can write $P = \bigcup_{\alpha \in \Lambda} P_\alpha$ where $(P_\alpha)_{\alpha \in \Lambda}$ for a directed set of countably generated direct summands of P . By Proposition 2.12, $I = \text{Tr}(P) = \sum_{\alpha \in \Lambda} \text{Tr}(P_\alpha)$ is a directed union of pure ideals of R , so that it is a pure ideal of R . \square

Corollary 2.14. *Let R be a commutative ring such that all pure ideals of $R/J(R)$ are finitely generated (hence generated by an idempotent). Then all pure ideals of R are of the form Re for $e = e^2 \in R$.*

Proof. Let I be a countably generated pure ideal of R . By [Remark 2.10](#), $I = \sum_{n \in \mathbb{N}} a_n R$ with $a_{n+1} a_n = a_n$ for any $n \in \mathbb{N}$. By hypothesis, there exists $n_0 \in \mathbb{N}$, such that $a_{n_0} R + J(R) = a_{n_0+k} R + J(R)$ for any $k \in \mathbb{N}$. By [Corollary 2.8](#), $I = a_{n_0} R$. Since R/I is a finitely presented flat module (or using the determinant trick) we deduce that I and R/I are projective, so that, I is generated by an idempotent.

In particular, an ascending chain

$$e_1 R \subseteq e_2 R \subseteq \dots \subseteq e_n R \subseteq \dots$$

where, for any $n \geq 1$, $e_n^2 = e_n \in R$ must be stationary because $I = \bigcup_{n \geq 1} e_n R$ is a pure ideal of R .

If I is any pure ideal of R then, as pointed out in [Remark 2.10](#), the set of countably generated pure ideals contained in I form a direct system, closed under countable unions and with limit I . By the first part of the proof, I is generated by idempotents. Since any ascending chain of principal ideals generated by an idempotent must terminate, we deduce that I is generated by a single idempotent. \square

3. Lifting projective modules modulo a trace ideal

Let I be an ideal of R that is the trace of a projective right module. Let $J+I$ be an ideal of R/I that is the trace ideal of a projective right R/I -module. Using the characterization in [Proposition 2.6](#), it is not difficult to see that $I + J$ is also a trace ideal of a projective right module over R . We will see the stronger result that a projective module modulo a trace ideal can be lifted to a projective module over R . The main step will be to prove the result in the countably generated case, to do that we extend the ideas of [\[13, Lemma 2.6\]](#) which, in turn, are based in Whitehead’s method to construct countably generated projective modules with prescribed trace ideal; cf. [\[22\]](#) or [Lemma 1.3](#).

Theorem 3.1. *Let R be a ring, and let I be an ideal of R that is the trace of a countably generated projective right R -module. Let P' be a countably generated projective right module over R/I , then there exists a countably generated projective right R -module P such that $I \subseteq \text{Tr}(P)$, $P/PI \cong P'$ and $\text{Tr}(P)/I = \text{Tr}_{R/I}(P')$.*

Proof. According to [Proposition 1.4](#), we can choose $\{X_k\}_{k \geq 1}$ and $\{Y_k\}_{k \geq 1}$ two sequences of finite matrices with entries in R such that they define a direct system of finitely generated free R/I -modules

$$(R/I)^{n_1} \xrightarrow{\overline{X_1}} (R/I)^{n_2} \xrightarrow{\overline{X_2}} \dots (R/I)^{n_k} \xrightarrow{\overline{X_k}} (R/I)^{n_{k+1}} \dots$$

with limit P' and such that $Y_k X_{k+1} X_k - X_k \in M_{n_{k+1} \times n_k}(I)$ for any $k \geq 1$. To avoid trivial situations we assume $n_k \geq 1$ for any $k \geq 1$.

By [Proposition 2.4\(iii\)](#) and [Lemma 2.2](#), $I = \bigcup_{k \geq 1} J_k R$ where $\{J_k\}_{k \geq 1}$ is an ascending chain of finitely generated left ideals of R such that, for each $k \geq 1$, $Y_k X_{k+1} X_k - X_k \in$

$M_{n_{k+1} \times n_k}(J_k)$ and there exists a nonempty finite set $\{a_1^k, \dots, a_{\ell_k}^k\}$ of generators of J_k such that $J_k = \sum_{i=1}^{\ell_k} J_{k+1} a_i^k$. For any $k \geq 1$, let $a_k = \begin{pmatrix} a_1^k \\ \vdots \\ a_{\ell_k}^k \end{pmatrix}$.

We define inductively a sequence of natural numbers $\{m_k\}_{k \geq 1}$ by setting $m_1 = n_1$ and, for $k \geq 1$, $m_{k+1} = n_{k+1} + m_k \ell_k$. Now we want to construct a direct limit of finitely generated free modules

$$R^{m_1} \xrightarrow{A_1} R^{m_2} \xrightarrow{A_2} \dots R^{m_k} \xrightarrow{A_k} R^{m_{k+1}} \dots$$

such that $\varinjlim (R^{m_k}, A_k) = P$ is a projective module and $P \otimes R/I \cong P'$. To this aim and for each $k \geq 1$ we name the following auxiliary matrices,

$$A'_k = \begin{pmatrix} a_k & \dots & 0 \\ & \ddots & m_k \\ 0 & \dots & a_k \end{pmatrix} \in M_{(m_k \ell_k) \times m_k}(R) \quad \text{and} \quad X'_k = \begin{pmatrix} X_k & 0 \end{pmatrix} \in M_{n_{k+1} \times m_k}(R).$$

Now we set, for each $k \geq 1$, $A_k = \begin{pmatrix} X'_k \\ A'_k \end{pmatrix} \in M_{m_{k+1} \times m_k}(R)$. Observe that

$$\varinjlim (R^{m_k}, A_k) \otimes_R R/I \cong \varinjlim \left((R/I)^{m_k}, \begin{pmatrix} \overline{X_k} \\ 0 \end{pmatrix} \right) \cong P'.$$

By Proposition 1.4, to conclude that P is a projective module, we need to find a sequence of square matrices $\{C_k\}_{k \geq 1}$ such that $C_k \in M_{m_{(k+1)} \times m_{(k+2)}}(R) A_{k+1}$ and $C_k A_k = A_k$ for any $k \geq 1$.

Since, by construction, $Z_k = Y_k X_{k+1} X'_k - X'_k \in M_{n_{k+1} \times m_k}(J_k)$, by Lemma 2.3, $Z_k = C_1^k A'_k$ for some $C_1^k \in M_{n_{k+1} \times m_k \ell_k}(J_{k+1})$. Also by Lemma 2.3, there exists $C_2^k \in M_{m_k \ell_k}(J_{k+1})$ such that $C_2^k A'_k = A'_k$. Then, for any $k \geq 1$, the matrix

$$C_k = \begin{pmatrix} Y_k X_{k+1} & -C_1^k \\ 0 & C_2^k \end{pmatrix}$$

satisfies that $C_k A_k = A_k$.

Write $A_{k+1} = \begin{pmatrix} X_{k+1} & 0 \\ B_{k+1} & B'_{k+1} \end{pmatrix}$. Let $D_2^k \in M_{m_k \ell_k \times m_{k+1} \ell_{k+1}}(R)$ be such that $(0 \ C_2^k) = D_2^k A'_{k+1}$. Further, using Lemma 2.3 we find $D_1^k \in M_{n_{k+1} \times m_{k+1} \ell_{k+1}}(R)$ such that $D_1^k B_{k+1} = 0$ and $D_1^k B'_{k+1} = -C_1^k$. Then

$$C_k = \begin{pmatrix} Y_k & D_1^k \\ 0 & D_2^k \end{pmatrix} A_{k+1}.$$

Hence P is a projective module as claimed.

Finally, by Proposition 1.4 and the construction of P , it also follows that $\text{Tr}(P)/I = \text{Tr}_{R/I}(P')$. \square

Corollary 3.2. *Let R be a ring, and let I be an ideal of R that is the trace of a projective right R -module. Let P' be a projective right module over R/I , then there exists a projective right R -module P such that $I \subseteq \text{Tr}(P)$, $P/PI \cong P'$ and $\text{Tr}(P)/I = \text{Tr}_{R/I}(P')$.*

Proof. We may assume that P' is a countably generated projective right R/I -module. By Proposition 1.4, we can choose $\{X_k\}_{k \geq 1}$ and $\{Y_k\}_{k \geq 1}$ two sequences of finite matrices with entries in R such that they define a direct system of finitely generated free R/I -modules

$$(R/I)^{n_1} \xrightarrow{\overline{X_1}} (R/I)^{n_2} \xrightarrow{\overline{X_2}} \dots (R/I)^{n_k} \xrightarrow{\overline{X_k}} (R/I)^{n_{k+1}} \dots$$

with limit P' and such that $Y_k X_{k+1} X_k - X_k \in M_{n_{k+1} \times n_k}(I)$ for any $k \geq 1$.

By Proposition 2.6, there exists $I_C \subseteq I$ such that I_C is the trace of a countably generated projective right R -module, and I_C contains the entries of the matrices $(Y_k X_{k+1} X_k - X_k)_{k \geq 1}$. Consider the directed system of free R/I_C -modules,

$$(R/I_C)^{n_1} \xrightarrow{\overline{X_1}} (R/I_C)^{n_2} \xrightarrow{\overline{X_2}} \dots (R/I_C)^{n_k} \xrightarrow{\overline{X_k}} (R/I_C)^{n_{k+1}} \dots$$

By Proposition 1.4, its direct limit is a countably generated projective right R/I_C -module, P'' say. By Theorem 3.1, there exists P a countably generated projective right R -module such that $\text{Tr}(P) \supseteq I_C$ and $P/PI_C \cong P''$.

Let Q be a projective right R -module such that $\text{Tr}(Q) = I$. Then $P \oplus Q$ satisfies the desired properties. \square

4. An application: projective modules over semi-semiperfect rings

A ring R is said to be semilocal if $R/J(R)$ is semisimple artinian. It is said to be semi-semiperfect if it is semilocal and for any simple right R -module V there exists $n_V \in \mathbb{N}$ and a finitely generated projective right R -module P such that $P/PJ(R) \cong V^{n_V}$. Observe that, if for any simple right R -module V , n_V can be taken to be 1 then R is a semiperfect ring.

Semi-semiperfect rings are closed by quotients by two-sided ideals. For further quoting we record this fact in the next lemma.

Lemma 4.1. *Let R be a semilocal ring, and let $I \neq R$ be a two-sided ideal of R . If R is semi-semiperfect then so is R/I .*

Let R be a ring. We denote by $V(R)$ the (commutative) monoid of isomorphism classes of finitely generated projective right R -modules with the addition induced by the direct sum of projective modules. We denote by $V^*(R) = V^*(R_R)$ the (commutative) monoid of isomorphism classes of countably generated projective right R -modules with the addition induced by the direct sum of projective modules.

We note the following corollary of Theorem 3.1.

Corollary 4.2. *Let R be a ring and let I be the trace ideal of a countably generated projective module. Then the assignment $\langle P \rangle \mapsto \langle P/PI \rangle$ for $\langle P \rangle \in V^*(R)$ defines an onto morphisms of monoids $V^*(R) \rightarrow V^*(R/I)$.*

As an application of the results in the previous sections we will show that if R is a semi-semiperfect ring then the monoids $V^*(R)$ are finitely generated and, in fact, are included in the class of monoids that can be realized as $V^*(R)$ for R a semilocal noetherian ring.

We introduce some notation in order to be able to state and prove our results.

Our convention is $\mathbb{N} = \{1, 2, \dots\}$, and we denote the nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We also need the monoid $(\mathbb{N}_0^*, +, 0)$ whose underlying set is $\mathbb{N}_0 \cup \{\infty\}$, the operation $+$ is the extension of addition of non-negative integers by the rule $\infty + x = x + \infty = \infty$. It is also interesting to extend the product of \mathbb{N}_0 to \mathbb{N}_0^* by setting $\infty \cdot 0 = 0$ and $\infty \cdot n = \infty$ for any $n \in \mathbb{N}_0^* \setminus \{0\}$.

Let R be a semilocal ring such that $R/J(R) \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$ for suitable division rings D_1, \dots, D_k . Fix an onto ring homomorphism $\varphi: R \rightarrow M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$ such that $\text{Ker } \varphi = J(R)$.

In this situation, the monoids $V(R)$ and $V^*(R)$ can be seen a submonoids of \mathbb{N}_0^k and of $(\mathbb{N}_0^*)^k$, respectively. We briefly explain how this can be made. We will keep using the fact that two projective right R -modules are isomorphic if and only if they are isomorphic modulo $J(R)$ [14].

Fix V_1, \dots, V_k to be an ordered set of representatives of the isomorphism classes of simple right R -modules. For $i = 1, \dots, k$, we assume that $\text{End}_R(V_i) = D_i$. For any countably generated projective right R -module P , $P/PJ(R) \cong V_1^{(I_1)} \oplus \dots \oplus V_k^{(I_k)}$. Denote by $\langle P \rangle$ the isomorphism class of P , then we set

$$\dim_\varphi(\langle P \rangle) := (m_1, \dots, m_k) \in (\mathbb{N}_0^*)^k,$$

where, for $i = 1, \dots, k$, $m_i = |I_i|$ if I_i is finite and $m_i = \infty$ if I_i is infinite. Since $P \cong Q$ if and only if $P/PJ(R) \cong Q/QJ(R)$ [14], the map $\dim_\varphi: V^*(R) \rightarrow (\mathbb{N}_0^*)^k$ is an injective morphism of additive monoids.

Notice that $\dim_\varphi(V(R)) \subseteq \mathbb{N}_0^k$. Moreover, in this case the monoid morphism \dim_φ is an embedding of monoids (that is, if in \mathbb{N}_0^k there is an equality $x + y = z$ with $x, z \in \dim_\varphi(V(R))$ then also $y \in \dim_\varphi(V(R))$). It was proved in [6] that this property characterizes the monoids that can be realized as $V(R)$ for a semilocal ring R .

The following result characterizes semilocal rings that are semi-semiperfect in terms of the values of \dim_φ and also shows that to test that a projective module is finitely generated it is enough to show that it is finitely generated modulo the Jacobson radical.

Lemma 4.3. *With the notation above. The ring R is semi-semiperfect if and only if, for $i = 1, \dots, k$, there exists $m_i \in \mathbb{N}$ such that $(0, \dots, m_i^i, \dots, 0) \in \dim_\varphi(V(R))$.*

Moreover, in this situation, if P_R is a countably generated projective right R -module such that $P/PJ(R)$ is finitely generated then P_R is finitely generated.

Proof. The first part of the statement is just a translation of the definition of semi-semiperfect ring to the monoid $V(R)$.

To prove the second part, let P_R be a countably generated projective right R -module such that $P/PJ(R)$ is finitely generated. Hence there exists $(a_1, \dots, a_k) \in \mathbb{N}_0^k$ such that $\dim_\varphi(\langle P \rangle) = (a_1, \dots, a_k)$. For $i = 1, \dots, k$, let P_i be a finitely generated right projective module such that $\dim_\varphi(\langle P_i \rangle) = (0, \dots, m_i^i, \dots, 0)$ and set $t_i = \prod_{j \neq i} m_j$. Therefore

$$\dim_\varphi(\langle P^{m_1 \cdots m_k} \rangle) = \dim_\varphi \left(\left\langle \bigoplus_{i=1}^k P_i^{a_i t_i} \right\rangle \right)$$

so that $P^{m_1 \cdots m_k} \cong \bigoplus_{i=1}^k P_i^{a_i t_i}$ is finitely generated and, hence, we conclude that P is a finitely generated projective right R -module. \square

Let $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{N}_0^*)^k$. We define

$$\text{supp}(\mathbf{x}) = \{i \in \{1, \dots, k\} \mid x_i \neq 0\}$$

and we refer to this set as the *support of \mathbf{x}* . We also define

$$\text{inf-supp}(\mathbf{x}) = \{i \in \{1, \dots, k\} \mid x_i = \infty\},$$

we refer to this set as the *infinite support of \mathbf{x}* .

In the next lemma we recall that if P is a projective right R -module then the trace ideal records the support of $\dim_\varphi(\langle P \rangle)$.

Lemma 4.4. (See [8, Lemma 2.2].) *Let P be a countably generated projective right R -module with trace ideal I . Set $\mathbf{x} = \dim_\varphi(\langle P \rangle)$. For $i = 1, \dots, k$, the following statements are equivalent:*

- (i) V_i is a quotient of P .
- (ii) V_i is a quotient of I .
- (iii) $I + r_R(V_i) = R$.
- (iv) $i \in \text{supp}(\mathbf{x})$.

In view of Lemma 4.4 there is an injective map between

$$\Phi: \mathcal{T} = \{I \leq R \mid I \text{ is a trace ideal of a projective right module}\} \rightarrow \mathcal{P}(\{1, \dots, k\})$$

given by $\Phi(I) = \text{supp } \dim_\varphi(\langle P \rangle)$ where P_R is any (countably) generated projective right R -module with trace ideal I . Notice that if R is semi-semiperfect then Φ is a bijective map.

Theorem 4.5. *Let R be a semi-semiperfect ring. For $i = 1, \dots, k$, let P_i be a finitely generated projective right R -module such that $\Phi(\text{Tr}(P_i)) = \{i\}$. For any $J \in \mathcal{T}$ fix G_J a finite set of projective right R -modules such that $\overline{G}_J = \{Q/QJ \mid Q \in G_J\}$ generates $V(R/J)$.*

Let P be a countably generated projective right R -module, let $X = \text{inf-sup}(\text{dim}_\varphi(\langle P \rangle))$ and let I be a trace ideal such that $\Phi(I) = X$ then

$$P \cong \left(\bigoplus_{i \in X} P_i^{(\omega_i)} \right) \oplus \left(\bigoplus_{Q \in G_I} Q^{n_Q} \right)$$

where, for any $Q \in G_I$, $n_Q \in \mathbb{N}_0$.

In particular, $V^(R)$ is finitely generated by $P_1^{(\omega_1)}, \dots, P_k^{(\omega_k)}$ and $\bigcup_{I \in \mathcal{T}} G_I$.*

Proof. The statement is trivial if $I = R$, so we may assume that $I \neq R$. By Lemma 4.1, R/I is a semi-semiperfect ring. Let dim_{φ_I} be the dimension function induced by dim_φ over R/I . Such dimension function satisfies that for any countably generated projective right R -module K

$$\text{dim}_{\varphi_I}(\langle K/KI \rangle) = \pi_X(\text{dim}_\varphi(\langle K \rangle))$$

Where $\pi_X: (\mathbb{N}_0^*)^k \rightarrow (\mathbb{N}_0^*)^{\{1, \dots, k\} \setminus X}$ denotes the canonical projection.

By Lemma 4.4 and because of the definition of I , $\text{dim}_{\varphi_I}(\langle P/PI \rangle)$ has all its components in \mathbb{N}_0 so that, by Lemma 4.3, P/PI is a finitely generated projective right R/I module. Therefore $P/PI \cong \bigoplus_{Q \in G_I} (Q/QI)^{n_Q}$ for suitable $n_Q \in \mathbb{N}_0$. Now it follows that

$$P \cong K = \left(\bigoplus_{i \in X} P_i^{(\omega_i)} \right) \oplus \left(\bigoplus_{Q \in G_I} Q^{n_Q} \right)$$

because $\text{dim}_\varphi(\langle P \rangle) = \text{dim}_\varphi(\langle K \rangle)$, so that P and K are isomorphic modulo $J(R)$, hence, they are isomorphic. \square

Remark 4.6. Notice that the crucial fact in Theorem 4.5 is that any infinite support of a countably generated projective R -module is the support of a projective module. A result in the same spirit of Theorem 4.5 could be obtained with a semilocal ring R such that for any simple right R -module V there exists a projective right module P_V such that $P_V/P_V J(R) \cong V^{(I)}$ for a suitable, possibly infinite, non-empty set I . If $R/J(R) \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$ for suitable division rings D_1, \dots, D_k , then we would also have that the map $\Phi: \mathcal{T} \rightarrow \mathcal{P}(\{1, \dots, k\})$ is onto. However instead of lifting to $V^*(R)$ the generators of $V(R/I)$ for any $I \in \mathcal{T}$ we would need to lift the generators of $W(R/I)$ to obtain a set of generators of the monoid $V^*(R)$.

We recall that for a semilocal ring R , $W(R)$ is the submonoid of $V^*(R)$ consisting of the isomorphism classes of countably generated projective right R -modules that are finitely generated modulo $J(R)$. As stated in [Lemma 4.3](#), for a semi-semiperfect ring $W(R) = V(R)$ but this is not true in the more general setting that we are describing, see [\[9, Example 3.6\]](#).

It is not difficult to see that the monoids described in [Theorem 4.5](#) are a particular instance of the monoids given by a system of supports in the sense of [\[8, §7\]](#). Therefore, they can be described as the solutions in \mathbb{N}_0^* of suitable systems of linear equations and linear congruences with coefficients in \mathbb{N}_0 [\[8, Theorem 7.7\]](#). Now we use this technology to give some examples that illustrate that for semi-semiperfect rings not all projective modules are a direct sum of finitely generated ones.

Example 4.7. Let R be a semi-semiperfect ring and we fix an onto ring homomorphism $\varphi: R \rightarrow M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with kernel $J(R)$ and such that D_1, \dots, D_k are division rings. Then, by [\[8, Proposition 6.2\]](#), there exist $0 \leq n \leq k$, $A \in M_{n \times k}(\mathbb{N}_0)$ and m_1, \dots, m_n integers strictly bigger than one such that $\mathbf{x} \in \dim_\varphi(V(R))$ if and only if

$$A \cdot \mathbf{x}^t \in \begin{pmatrix} m_1 \mathbb{N}_0 \\ \vdots \\ m_n \mathbb{N}_0 \end{pmatrix} \tag{1}$$

By [\[8, Theorem 2.6\]](#), R can be chosen such that it is noetherian semilocal and $\mathbf{x} \in \dim_\varphi(V^*(R))$ if and only if

$$A \cdot \mathbf{x}^t \in \begin{pmatrix} m_1 \mathbb{N}_0^* \\ \vdots \\ m_n \mathbb{N}_0^* \end{pmatrix} \tag{2}$$

We will use this terminology to describe our example.

Let R be a (noetherian) semilocal ring such that $M = \dim_\varphi(V^*(R))$ is the set of solutions of the system given by the two congruences

$$2x + 3y \in 5\mathbb{N}_0^*; \quad x + 2y \in 2\mathbb{N}_0^*$$

Since $(10, 0)$ and $(0, 5) \in M$, R is semi-semiperfect. Then $V(R)$ is generated by $G = \{(2, 2), (6, 1), (10, 0), (0, 5)\}$, and $V^*(R)$ is generated by $G \cup \{(\infty, 0), (0, \infty), (1, \infty)\}$. Notice that a projective right R -module P such that $\dim_\varphi(\langle P \rangle) = (1, \infty)$ is not a direct sum of finitely generated modules but so is P^2 .

5. Fair sized projective modules

Let R be a ring, and let M be a right R -module. Consider the set

$$\mathcal{I}(M) = \{I \trianglelefteq R \mid M/MI \text{ is finitely generated}\}$$

Note that $R \in \mathcal{I}(M)$, and that $\{0\} \in \mathcal{I}(M)$ if and only if M is finitely generated.

In general, if $I \in \mathcal{I}(M)$ and J is a two-sided ideal of R such that $I \subseteq J$ then $J \in \mathcal{I}(M)$.

When M is a countably generated projective module there is a way to produce elements of $\mathcal{I}(M)$ using an idempotent matrix defining M .

Lemma 5.1. *Let R be a ring let $A = (a_{i,j})$ be a countable column finite matrix with entries in R such that $A^2 = A$, and set $P = AR^{(\mathbb{N})}$. For any $k \geq 0$, let $I_k = \sum_{i>k, j \in \mathbb{N}} a_{ij}R$. Then,*

- (i) $\text{Tr}(P) = RI_0$.
- (ii) For any $k \geq 0$ and any $0 < i \leq k$,

$$\sum_{j \in \mathbb{N}} a_{ij}R + RI_k/RI_k = \sum_{1 \leq \ell \leq k} a_{i\ell}R + RI_k/RI_k.$$

In particular if $0 \leq n \leq k$, $I_n + RI_k/RI_k$ is finitely generated.

- (iii) For any $k \geq 0$, $RI_k \in \mathcal{I}(P)$.
- (iv) For any $k \geq 0$, there exists $n_k > k$ such that $I_{n_k}I_k = I_{n_k}$.

Proof. Statement (i) is clear. To prove (ii) note that, since A is an idempotent matrix, for any $j \in \mathbb{N}$

$$a_{ij} = \sum_{\ell \in \mathbb{N}} a_{i\ell}a_{\ell j} \in \sum_{\ell \leq k} a_{i\ell}a_{\ell j} + RI_k \subseteq \sum_{1 \leq \ell \leq k} a_{i\ell}R + RI_k/RI_k.$$

(iii). The projective module P is generated by the columns of A . The R/RI_k projective module P/PI_k is isomorphic to the module generated by the columns of the matrix $(a_{ij} + RI_k)$ which is an idempotent matrix with entries in R/RI_k such that only the first k rows may be different from zero. So P/PI_k is a direct summand of $(R/RI_k)^k$, therefore it is a finitely generated projective module.

(iv). Since the matrix A is column finite, for any given k there exists $n_k > k$ such that $a_{ij} = 0$ for $i \geq n_k$ and $j \in \{1, \dots, k\}$.

Now if a_{ij} is such that $i > n_k$ then either $a_{ij} = 0$ if $j \in \{1, \dots, k\}$ or

$$a_{ij} = \sum_{\ell > k} a_{i\ell}a_{\ell j} \in I_{n_k}I_k.$$

This implies that $I_{n_k}I_k = I_{n_k}$. \square

Lemma 5.2. (See [13, Lemma 2.4].) *Let R be a ring. Let P be a countably generated projective right R -module. Let $A = (a_{ij})$ be a countable column finite idempotent matrix such that $P \cong AR^{(\mathbb{N})}$. For any $k \in \mathbb{N}$ set $I_k = \sum_{j \in \mathbb{N}, i > k} a_{ij}R$. Then*

- (i) *For any $I \in \mathcal{I}(P)$, there exists $k \in \mathbb{N}$ such that $RI_k \subseteq I$.*
- (ii) *$\mathcal{I}(P)$ is closed under finite intersections.*
- (iii) *If $\mathcal{I}(P)$ has minimal elements then it has a unique minimal element.*
- (iv) *$\mathcal{I}(P)$ has minimal elements if and only if there exists $k_0 \in \mathbb{N}$ such that, for any $\ell \in \mathbb{N}$, $RI_{k_0} = RI_{k_0+\ell}$. In this case $I_{k_0} = I_{k_0}^2$ and RI_{k_0} is the minimal element of $\mathcal{I}(P)$.*

Proof. (i). Since $AR^{(\mathbb{N})}$ is generated by the columns C_1, \dots, C_n, \dots of the matrix A , if $\bar{P} = AR^{(\mathbb{N})}/AI^{(\mathbb{N})}$ is finitely generated there exists n such that $S = \{C_1 + AI^{(\mathbb{N})}, \dots, C_n + AI^{(\mathbb{N})}\}$ generate \bar{P} . As A is column finite, there exists k such that, $a_{ij} = 0$ for $j \in \{1, \dots, n\}$ and $i > k$. Since for any $\ell > n$, $C_\ell + AI^{(\mathbb{N})}$ is an R -linear combination of the elements in S , we deduce that $a_{ij} \in I$ for any $j \in \mathbb{N}$ and any $i > k$. So that $RI_k \subseteq I$.

Statement (ii) follows from (i) and from the fact that, for any k , $RI_k \in \mathcal{I}(P)$. Statement (iii) follows from (ii), and (iv) is a consequence of (ii) and Lemma 5.1(iv). \square

Following the idea of [5, Proof of Proposition 4.2] we can prove further closure properties of the set $\mathcal{I}(P)$.

Lemma 5.3. *Let R be a ring and let P be a right R -module and let P_0 be a submodule of P . Let I be a two-sided ideal of R . Then*

- (i) *$P = P_0 + PI$ if and only if, for any right R/I -module M , $\text{Hom}_R(P/P_0, M) = 0$.*
- (ii) *Assume P is projective and P_0 is finitely generated then if $\text{Hom}_R(P/P_0, R/I) = 0$ then P/PI is a finitely generated projective right R -module.*

Proof. To prove both statements we have to keep in mind the following particular case of the Hom- \otimes -adjunction: for any right R/I -module M

$$\begin{aligned} \text{Hom}_R(P/P_0, M) &\cong \text{Hom}_R(P/P_0, \text{Hom}_{R/I}(R/I, M)) \cong \text{Hom}_{R/I}(P/P_0 \otimes_R R/I, M) \\ &\cong \text{Hom}_{R/I}(P/(P_0 + PI), M) \end{aligned}$$

Then (i) is clear. To prove (ii), notice that the assumption and the above formula imply that

$$\text{Hom}_{R/I}(P/(P_0 + PI), R/I) = 0.$$

Notice that $P/(P_0 + PI) \cong (P/PI)/(P_0 + PI/PI)$ and that $P_0 + PI/PI$ is a finitely generated submodule of the projective R/I -module P/PI . Now we follow the argument of [5, Lemma 4.1] to conclude.

There is a set X and a splitting embedding $f: P/PI \rightarrow (R/I)^{(X)}$; since $P_0 + PI/PI$ is finitely generated, there exists a finite subset Y of X such that $f(P_0 + PI/PI) \subseteq (R/I)^Y$. For any $x \in X$, let $p_x: (R/I)^{(X)} \rightarrow R/I$ denote the projection onto the x -component. The hypothesis implies that $p_x f = 0$ for any $x \in X \setminus Y$, hence P/PI is isomorphic to a direct summand of $(R/I)^Y$ and, hence, it is finitely generated. \square

Proposition 5.4. *Let R be a ring and let P be a projective right R -module. Fix a finitely generated submodule P_0 of P and set*

$$\mathcal{I}(P_0, P) = \{I \text{ ideal of } R \mid P = P_0 + PI\}.$$

Then

- (1) $\mathcal{I}(P_0, P)$ is closed under products.
- (2) If $\{I_i\}_{i \in \Lambda}$ is a family of ideals in $\mathcal{I}(P_0, P)$ then $\bigcap_{i \in \Lambda} I_i \in \mathcal{I}(P)$.
- (3) If $I \in \mathcal{I}(P)$ then $I^\omega = \bigcap_{n \geq 1} I^n \in \mathcal{I}(P)$.

Proof. (1). Let I, J be ideals in $\mathcal{I}(P_0, P)$. In view of Lemma 5.3(i), we must show that $\text{Hom}_R(P/P_0, M) = 0$ for any right R/IJ module M . To this aim we apply the functor $\text{Hom}_R(P/P_0, -)$ to the short exact sequence

$$0 \rightarrow MI \rightarrow M \rightarrow M/MI \rightarrow 0$$

to obtain the exact sequence

$$0 \rightarrow \text{Hom}_R(P/P_0, MI) \rightarrow \text{Hom}_R(P/P_0, M) \rightarrow \text{Hom}_R(P/P_0, M/MI).$$

Since MI is a right R/I -module, $\text{Hom}_R(P/P_0, MI) = 0$ and, since M/MI is a right R/I -module, $\text{Hom}_R(P/P_0, M/MI) = 0$. Hence $\text{Hom}_R(P/P_0, M) = 0$ as wanted.

(2). Applying the functor $\text{Hom}_R(P/P_0, -)$ to the exact sequence

$$0 \rightarrow R / \bigcap_{i \in \Lambda} I_i \rightarrow \prod_{i \in \Lambda} R/I_i$$

we obtain

$$\begin{aligned} 0 \rightarrow \text{Hom}_R\left(P/P_0, R / \bigcap_{i \in \Lambda} I_i\right) &\rightarrow \text{Hom}_R\left(P/P_0, \prod_{i \in \Lambda} R/I_i\right) \\ &\cong \prod_{i \in \Lambda} \text{Hom}_R(P/P_0, R/I_i) = 0. \end{aligned}$$

Therefore, $\text{Hom}_R(P/P_0, R / \bigcap_{i \in \Lambda} I_i) = 0$ and then the statement follows from Lemma 5.3(ii).

Statement (3) follows from (1) and (2). \square

Definition 5.5. (See [13].) Let R be a ring. A countably generated projective right R -module P is *fair sized* if $\mathcal{I}(P)$ has a minimal element.

Proposition 5.6. *Let R be a semilocal ring, and let P be a countably generated projective right R -module then P is fair sized.*

Proof. We can assume that P is not finitely generated and $\mathcal{I}(P) \neq \{R\}$.

Let $\mathcal{S} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ denote the set of all maximal two-sided ideals of R . If $R \neq I \in \mathcal{I}(P)$, then there exists $i \in \{1, \dots, k\}$ such that $I \subseteq \mathcal{M}_i$. In particular, $\mathcal{M}_i \in \mathcal{I}(P)$.

Reindexing the elements of \mathcal{S} , if necessary, we may assume that there exists $r \geq 1$ such that $P/P\mathcal{M}_i$ is finitely generated if $i \leq r$ and $P/P\mathcal{M}_i$ is infinitely generated for $i > r$. By Lemma 5.2(ii), $J = \mathcal{M}_1 \cap \dots \cap \mathcal{M}_r \in \mathcal{I}(P)$.

Fix $A = (a_{ij})$ a column finite idempotent matrix such that $P \cong AR^{(\mathbb{N})}$. For $k > 0$, set $I_k = \sum_{i < k, j \in \mathbb{N}} a_{ij}R$. By Lemma 5.2(i), there exists k_0 such that for any $\ell \geq k_0$, $RI_\ell \subseteq J$. Moreover, in view of Lemma 5.1 and the above remarks, for any $\ell \geq k_0$, $J(R/RI_\ell) = J/RI_\ell$.

By Lemma 5.1, there exists $k'_0 > k_0$ such that $I_{k'_0}I_{k_0} = I_{k'_0}$. Let $\ell > k'_0$. Then

$$M = I_{k'_0} + RI_\ell/RI_\ell = (I_{k'_0} + RI_\ell/RI_\ell)(I_{k_0} + RI_\ell/RI_\ell) = MJ(R/RI_\ell).$$

By Lemma 5.1, M is a finitely generated right R/RI_ℓ -module. By Nakayama's Lemma, $M = 0$ or, equivalently, $I_{k'_0} \subseteq RI_\ell$. In view of Lemma 5.2, we deduce that $\mathcal{I}(P)$ has a least element as we wanted to prove. \square

The first part of the next result is due to Sakhaev [20, Theorem 4]. We give an alternative proof following the same ideas used in the proof of Proposition 5.6.

Proposition 5.7. *Let R be a ring. Let P be a countably generated projective right R -module such that $P/PJ(R)$ is finitely generated. Then $\mathcal{I}(P)$ has a minimal element.*

In particular, if P is not finitely generated then $J(R)$ contains a non-zero idempotent ideal.

Proof. We may assume that P is not finitely generated. Fix an idempotent column finite matrix $A = (a_{ij})$ such that $P \cong AR^{(\mathbb{N})}$. For any $k \in \mathbb{N}$ set $I_k = \sum_{j \in \mathbb{N}, i < k} a_{ij}R$. Since $J(R) \in \mathcal{I}(P)$, by Lemma 5.2(i), there exists k_0 such that $RI_{k_0} \subseteq J(R)$. Now the statement follows using the same argument as in the proof of Proposition 5.6.

By Lemma 5.2, there exists $\ell \in \mathbb{N}$ such that the minimal element in $\mathcal{I}(P)$ is $RI_\ell \neq 0$ which, in addition, is an idempotent ideal. By the above argument $RI_\ell \subseteq J(R)$. \square

Since a projective module with a semilocal endomorphism ring is finitely generated modulo its Jacobson radical we deduce the following corollary.

Corollary 5.8. *Any projective module with a semilocal endomorphism ring is fair sized.*

Remark 5.9. The first example of a countably generated projective module, not finitely generated, but finitely generated modulo its Jacobson radical was given in [7]. Such example is, in fact, over a semilocal ring.

In [4], a detailed study of Gerasimov and Sakhaev example was made. In particular, it follows from [4, Theorem 6.8], that such a ring R has an idempotent ideal that it is trace ideal of a projective right R -module but it is not a trace ideal of a projective left R -module.

In [4, Question 5.6, Question 6.6] it was asked whether Gerasimov and Sakhaev’s example satisfies that $\bigcap_{n \geq 1} J(R)^n = \{0\}$ and whether such a ring has idempotent ideals contained in $J(R)$. Propositions 5.6 and 5.7 answer both questions in the negative. Notice that [5, Theorem C] gives also a negative answer to the first question.

Let R be a noetherian ring. Let P_1 and P_2 be countably generated, fair sized, projective right R -modules such that, for $i = 1, 2$, I is the minimal ideal such that $P_i/P_i I$ is finitely generated. Then $P_1 \cong P_2$ is and only if $P_1/P_1 I \cong P_2/P_2 I$ [13]. This result was the crucial tool to classify projective modules over some classes of noetherian rings: generalized Weyl algebras, semilocal noetherian rings, some lattices. However, as the following example shows, this is no longer true in general.

Example 5.10. Let R be a ring such that there is an isomorphism of monoids $\varphi: V(R) \rightarrow \mathbb{R}_0^+$ and such that $\varphi(\langle R \rangle) > \sum_{i \geq 1} \frac{1}{i^2}$. The existence of such ring is ensured by a theorem due to Bergman and Dicks [3].

For any $i \geq 1$, let P_i be a finitely generated projective right R -module such that $\varphi(\langle P \rangle) = \frac{1}{i^2}$. Then $R^{(\omega)}$ and $P = \bigoplus_{i \geq 1} P_i$ are uniformly big projective modules. That is, both are fair sized and the minimal ideal such that modulo it they are finitely generated is $I = R$. Hence $0 = R^{(\omega)}/R^{(\omega)}I = P/PI$ but P has no nonzero free direct summands. In particular $P \not\cong R^{(\omega)}$.

6. A particular way to think the reconstruction of the projective module

Let T be a countable rooted tree with no leaves. Let V be the set of vertices of T , let $v_0 \in V$ be the root and let E be the set of edges. Let $s: E \rightarrow V$ be the source map and $t: E \rightarrow V$ be the end map. That is, an edge e starts in $s(e)$ and terminates in $t(e)$. We assume that for each vertex v there is only a finite number of edges e such that $s(e) = v$.

Consider the distance function $d: V \times V \rightarrow \mathbb{N}_0$ where, by definition, $d(v, v')$ is the length of the shortest (unoriented) path from v to v' . For every $i \in \mathbb{N}_0$, set $V_i = \{v \in V \mid d(v, v_0) = i\}$, and let E_i be the set of edges connecting a vertex of V_i and a vertex of V_{i+1} .

Let R be a ring, and fix a representation of T in $\text{Mod-}R$. The representation assigns to each $v \in V$ a module M_v and to every $e \in E$ a homomorphism $f_e: M_{s(e)} \rightarrow M_{t(e)}$. Set $M_i = \bigoplus_{v \in V_i} M_v$. For every $v \in V_i$ let $\iota_v^i: M_v \rightarrow M_i$ be the canonical embedding and let $\pi_v^i: M_i \rightarrow M_v$ be the canonical projection. For every $i \in \mathbb{N}_0$ there is a unique

homomorphism $f_i: M_i \rightarrow M_{i+1}$ such that, for every $e \in E_i$, $\pi_{t(e)}^{i+1} f_i \iota_{s(e)}^i = f_e$. Therefore we get a direct system

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots M_k \xrightarrow{f_k} M_{k+1} \dots$$

whose limit fits into a pure exact sequence

$$0 \rightarrow \bigoplus_{v \in V} M_v \rightarrow \bigoplus_{v \in V} M_v \rightarrow \varinjlim M_i \rightarrow 0 \tag{3}$$

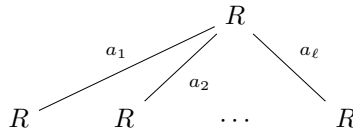
Now we introduce some further notation in order to be able to give a local criteria to ensure that this exact sequence splits.

For every $v \in V$ let us denote $S(v)$ the set of successors of v , that is $S(v) = \{v' \in V \mid \exists e \in E \text{ such that } s(e) = v, t(e) = v'\}$. Let $S_2(v) = \bigcup_{v' \in S(v)} S(v')$. For any vertex v let $M'_v = \bigoplus_{u \in S(v)} M_u$ and let $M''_v = \bigoplus_{u \in S_2(v)} M_u$. For every $u \in S(v)$ and $w \in S_2(v)$ let $\pi'_u: M'_v \rightarrow M_u$ and $\pi''_w: M''_v \rightarrow M_w$ be the canonical projections and let $\iota'_u: M_u \rightarrow M'_v$ be the canonical embedding. Then there exist a unique homomorphism $g_v: M_v \rightarrow M'_v$ such that $\pi'_{t(e)} g_v = f_e$ for any $e \in E$ such that $v = s(e)$. There is also a unique homomorphism $g'_v: M'_v \rightarrow M''_v$ such that $\pi''_{t(e')} g'_v \iota'_{s(e')} = f_{e'}$ for any $e' \in E$ satisfying that $s(e') \in S(v)$.

Lemma 6.1. *With the notation above. If, for every $v \in V$, there exists a homomorphism $h_v: M''_v \rightarrow M'_v$ such that $h_v g'_v g_v = g_v$ then the sequence (3) splits.*

Proof. Notice that, for every $i \in \mathbb{N}_0$, $M_{i+1} = \bigoplus_{v \in V_i} M'_v$ and $M_{i+2} = \bigoplus_{v \in V_i} M''_v$. Therefore, for every $i \in \mathbb{N}_0$, there is a homomorphism $g_{i+1}: M_{i+2} \rightarrow M_{i+1}$ given by $g_{i+1} = \bigoplus_{v \in V_i} h_v$. Using the condition $h_v g'_v g_v = g_v$ it is easy to see that $g_{i+1} f_{i+1} f_i = f_i$ holds. By Lemma 1.1, the sequence (3) splits. \square

Example 6.2. Suppose that I is a nonzero idempotent ideal of R generated as a left ideal by a_1, a_2, \dots, a_ℓ . Then a construction of a projective module having the trace ideal I can be explained as follows: Let T be a rooted tree such that, for every vertex v , $|S(v)| = n$. For every $v \in V$, set $M_v = R$ and, if $e_1, \dots, e_\ell \in E$ are the edges such that $s(e) = v$, set $f_{e_i}: R \rightarrow R$ to be left multiplication by a_i . So that, the representation can be visualized as a repetition at each vertex of the following basic tree of height 1:



so that, for each $v \in V$, $g_v: R \rightarrow R^\ell$ is given by $g_v(r) = \begin{pmatrix} a_1 \\ \vdots \\ a_\ell \end{pmatrix} r$ for any $r \in R$. Set $A = \begin{pmatrix} a_1 \\ \vdots \\ a_\ell \end{pmatrix}$ and $B = \begin{pmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A \end{pmatrix}$. Then $g'_v: R^\ell \rightarrow (R^\ell)^\ell$ is given by $g'_v(r_1, \dots, r_\ell) = B \begin{pmatrix} r_1 \\ \vdots \\ r_\ell \end{pmatrix}$.

By Lemma 2.3, there exists $C \in M_{\ell \times \ell^2}(R)$ such that $CBA = A$. Now the homomorphism $h_v: (R^\ell)^\ell \rightarrow R^\ell$ given by left multiplication by C satisfies that $h_v g'_v g_v = g_v$. Therefore, by Lemma 6.1, the limit of the direct system

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots M_k \xrightarrow{f_k} M_{k+1} \dots$$

where, for any $k \geq 0$, $M_k = R^{\ell^k}$, is a projective module P . By Proposition 1.4, the trace of P is the ideal I .

Observe that, for any $k \geq 0$,

$$P \cong \varinjlim_{i \geq k} M_i \cong \left(\varinjlim_{i \geq 0} M_i \right)^{\ell^k} = P^{\ell^k} \neq \{0\}.$$

In particular, if R is a right noetherian ring, it follows that such projective module P cannot have finite Goldie dimension and, hence, it is always I -big, that is, P is not finitely generated, and if J is a right ideal of R such that P/PJ is finitely generated then $P = PJ$ and, hence, $RJ \supseteq I$.

Our next construction will use the following lemma.

Lemma 6.3. *Let R be a ring, and let $I_1 \supseteq I_2 \supseteq I_3$ be two-sided ideals of R such that I_3 is idempotent and ${}_R I_3$ is finitely generated. Assume that R/I_3 is semisimple and, for $i \leq j$, let $e_{ij} \in R$ be such that $e_{ij} + I_j$ is a central idempotent of R/I_j generating I_i/I_j . Then,*

- (i) I_1 and I_2 are also idempotent ideals which are finitely generated as left R -modules.
- (ii) Let $G \in M_{m \times 1}(R)$ and $H \in M_{n \times 1}(R)$ be column matrices such that their entries generate ${}_R I_2$ and ${}_R I_3$, respectively. Set $A = \begin{pmatrix} e_{1,2} \\ G \end{pmatrix}$, and

$$B = \begin{pmatrix} Y & \dots & 0 \\ 0 & Z & \dots & 0 \\ & & m) & \\ & & \ddots & \\ & & & Z \end{pmatrix}$$

where $Y = \begin{pmatrix} e_{1,3} \\ H \end{pmatrix}$ and $Z = \begin{pmatrix} e_{2,3} \\ H \end{pmatrix}$. Then there exists a matrix C of suitable size and with entries in R such that $CBA = A$.

Proof. (i). This statement follows easily from the fact that ${}_R I_3$ is finitely generated and R/I_3 is semisimple artinian.

(ii). Since $e_{1,2}R \subseteq Re_{1,2} + I_2$ and $I_2 = \sum_{i=1}^m Rg_i$, where g_1, \dots, g_m denote the entries of G , it follows as in Example 6.2 that there exists $D \in M_{(m+1) \times (m+1)}(R)$ such that $DA = A$. Since $e_{1,2}^2 - e_{1,2} \in I_2$, D can be taken of the form $D = (A|D')$ where D' has entries in I_2 .

Since the entries in Y generate I_1 and the entries in Z generate I_2 , one can proceed as in [Lemma 2.3](#) to show that there exists a suitable matrix C such that $D = CB$. \square

We are also interested in the following variation of [Lemma 6.3](#).

Remark 6.4. In the same situation as in [Lemma 6.3](#), fix $c, d \in \mathbb{N}$. Let $A_c = \begin{pmatrix} e_{1,2} \\ \vdots \\ c \\ e_{1,2} \\ G \end{pmatrix}$. Let

$$B_{cd} = \begin{pmatrix} Y_c & & \dots & 0 \\ & \ddots & & \\ & & Y_c & \\ 0 & & Z_d & \dots & 0 \\ & & & \ddots & \\ & & & & Z_d \end{pmatrix}$$

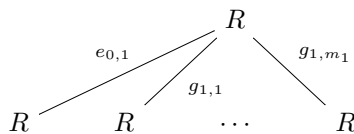
where $Y_c = \begin{pmatrix} e_{1,3} \\ \vdots \\ c \\ e_{1,3} \\ H \end{pmatrix}$ and $Z_d = \begin{pmatrix} e_{2,3} \\ \vdots \\ d \\ e_{2,3} \\ H \end{pmatrix}$. Then there exists a matrix C_{cd} of suitable size and with entries in R such that $C_{cd}B_{cd}A_c = A_c$.

From now on we assume that R is a ring having a descending chain of ideals $I_0 \supseteq I_1 \supseteq \dots$ such that

- 1) For every $i \in \mathbb{N}_0$ the ring R/I_i is semisimple artinian.
- 2) For every $i \in \mathbb{N}_0$ the ideal I_i is idempotent and RI_i is finitely generated.

For every $j \in \mathbb{N}$ let G_j be a finite set of generators RI_j , and for every $i, j \in \mathbb{N}_0, i < j$ let $e_{i,j}$ be chosen such that $e_{i,j} + I_j$ is a central idempotent of R/I_j generating I_i/I_j . Further let $m_i = |G_i|$ for every $i \in \mathbb{N}$.

Now we construct a tree T and a representation of T by R -modules. We proceed by induction on the distance from the root and we construct T and the representation simultaneously. We keep the notation from the beginning of the section. For every $v \in V$ we put $M_v = R$. The root v_0 of T has $1 + m_1$ successors. If e_0, \dots, e_{m_1} are the edges adjacent to v_0 then f_{e_0} is given by left multiplication by $e_{0,1}$, and $f_{e_1}, \dots, f_{e_{m_1}}$ are given, respectively, by left multiplication by the m_1 elements of G_1 . That is, if $G_1 = \{g_{1,1}, \dots, g_{1,m_1}\}$ the result of this first step can be visualized as



Now suppose we have constructed the subtree and the representation on the subtree T' given by the set of vertices $\bigcup_{0 \leq j \leq i} V_j$ for some $i \in \mathbb{N}$ in such a way that for $v \in V_i$:

- (i) there is a unique edge e_v in T' such that $v = t(e_v)$;
- (ii) f_{e_v} is given by left multiplication
 - (ii.a) left multiplication by $e_{j,i}$ for some $j < i$ or
 - (ii.b) by left multiplication by an element from G_i .

Now for $v \in V_i$, we add exactly $1 + m_{i+1}$ successors of v to T' and the corresponding edges say $e_0, \dots, e_{m_{i+1}}$. Then we set f_{e_0} to be given by left multiplication by $e_{j,i+1}$ in the situation (ii.a), otherwise we set $f(e_0)$ to be left multiplication by $e_{i,i+1}$. Further, we set $f_{e_1}, \dots, f_{e_{m_{i+1}}}$ to be given by left multiplication by the elements of G_{i+1} . Repeating this procedure for every $v \in V_i$ completes the inductive step.

By Lemma 6.3, for every $v \in V$ there exists $h_v: M''_v \rightarrow M'_v$ such that $h_v g'_v g_v = g_v$. Hence, by Lemma 6.1, the direct limit of the system $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots M_k \xrightarrow{f_k} M_{k+1} \dots$ given by the representation of T is a countably generated projective right module that we denote by P .

The module P satisfies the following properties.

Lemma 6.5.

- (i) The trace ideal of P is I_0 .
- (ii) For every $k \geq 0$,

$$P \otimes_R R/I_{k+1} \cong (I_0/I_{k+1}) \oplus (I_1/I_{k+1})^{m_1} \oplus (I_2/I_{k+1})^{(m_1+1)m_2} \oplus \dots \oplus (I_k/I_{k+1})^{(m_1+1)(m_2+1)\dots m_k}.$$

In particular, P/PI_k is finitely generated for any $k \geq 0$.

- (iii) For every $k \geq 0$, $P \cong P_{k0} \oplus \dots \oplus P_{kk}$, where $\text{Tr}(P_{ki}) = I_i$ for $i = 0, \dots, k$.
- (iv) If R_R is noetherian and, for any $k \geq 0$, $I_k \not\supseteq I_{k+1}$ then P is not fair sized.

Proof. Statement (i) is clear from the construction and Proposition 1.4 because the entries of the transition maps of the direct limit generate the ideal I_0 .

To prove statement (ii), note that

$$P/PI_{k+1} \cong P \otimes_R R/I_{k+1} \cong \varinjlim_{i \geq k} M_i \otimes_R R/I_{k+1} \cong \varinjlim_{i \geq k} M_i/M_i I_{k+1}.$$

The statement follows counting the number of semisimple factors in this direct limit.

Statement (iii) follows in a similar way, taking into account that the direct system $\{M_i\}_{i \geq k}$ is the direct sum of $k + 1$ direct systems, each one with limit a projective module. By Proposition 1.4 the traces of such projective modules are the ideals I_0, \dots, I_k , respectively.

(iv). Assume there is an ideal of R minimal with respect to the property that P/PI is finitely generated. Since R/I is a right noetherian ring, P/PI is a module with finite Goldie dimension n , say. In view of (iii), this implies that $P_{nn}/P_{nn}I = 0$. Equivalently, $I \supseteq I_n$. Then statement (ii) contradicts the minimality of I . \square

Making a variation of the construction of P we have the following extension of Lemma 6.5.

Lemma 6.6. *Let R be a ring. For any sequence s of elements in \mathbb{N} there exists a countably generated projective right R -module P_s with trace I_0 , and satisfying the following properties:*

(i) For every $k \geq 0$,

$$P_s \otimes_R R/I_{k+1} \cong (I_0/I_{k+1})^{\alpha_0} \oplus (I_1/I_{k+1})^{\alpha_1} \oplus \cdots \oplus (I_k/I_{k+1})^{\alpha_k},$$

where $\alpha_0, \dots, \alpha_k \in \mathbb{N}_0$ determine first $k + 1$ terms of the sequence s . In particular, P_s/P_sI_k is finitely generated for any $k \geq 0$.

(ii) If, for any $k \geq 0$, $I_k \not\supseteq I_{k+1}$ then $P_s \cong P_{s'}$ implies $s = s'$.

(iii) For every $k \geq 0$, $P_s \cong Q_{k0} \oplus \cdots \oplus Q_{kk}$, where $\text{Tr}(Q_{ki}) = I_i$ for $i = 0, \dots, k$.

(iv) If R_R is noetherian and, for any $k \geq 0$, $I_k \not\supseteq I_{k+1}$ then P_s is not fair-sized.

Proof. Let $c_0, c_1, \dots \in \mathbb{N}$ be the elements of the sequence s . Modify the construction before Lemma 6.5 as follows: Set $M_v = R$ for every $v \in V$, and let $a_{0,1}, \dots, a_{0,c_0}, e_1, \dots, e_{m_1}$ be the edges starting at v_0 . Then $f_{a_{0,i}}$ is given by left multiplication by $e_{0,1}$ and $f_{e_1}, \dots, f_{e_{m_1}}$ are given, respectively, by left multiplication by the m_1 elements of G_1 . For the inductive step, assume $i > 0$ and fix $v \in V_i$. Then $v = t(e_v)$ for a unique $e_v \in E$. Put $j = i$ provided f_{e_v} is given by multiplication by an element in G_i and put $j = k$ if f_{e_v} is given by left multiplication by $e_{k,i}$ for some $k < i$. If $j < i$ the vertex v has $1 + m_{i+1}$ successors, let us denote $e_0, e_1, \dots, e_{m_{i+1}}$ the edges starting at v . Then f_{e_0} is given by left multiplication by $e_{j,i+1}$ and $f_{e_1}, \dots, f_{e_{m_{i+1}}}$ are given by left multiplication by the different elements of G_{i+1} . If $i = j$ the vertex v has $c_i + m_{i+1}$ successors, let us denote $a_{i,1}, \dots, a_{i,c_i}, e_1, \dots, e_{m_{i+1}}$ the edges starting at v . Then $f_{a_{i,1}}, \dots, f_{a_{i,c_i}}$ are given by left multiplication by $e_{i,i+1}$ and $f_{e_1}, \dots, f_{e_{m_{i+1}}}$ are given by left multiplication by the elements from G_{i+1} .

Let P_s be the direct limit of the telescope constructed from this representation. Using Remark 6.4 and Lemma 6.1 it follows that P_s is countably generated projective. Proposition 1.4 ensures that the trace of P_s is I_0 .

Statement (i) follows as Lemma 6.5(ii). To prove statement (ii), let $c'_0, c'_1, \dots \in \mathbb{N}$ be the elements of a sequence s' . Notice that if $P_s \cong P_{s'}$ then, for any $k \geq 0$ there is an isomorphism between the finitely generated semisimple factors $S = (P_s/P_sI_{k+1})I_k$ and $S' = (P_{s'}/P_{s'}I_{k+1})I_k$. Therefore, by (i), $c_k = c'_k$ for any $k \geq 0$.

Statements (iii) and (iv) are proved as their counterparts in Lemma 6.5. \square

7. An application to FCR-algebras

In this section we present a context in which the constructions of the previous section can be done: the FCR-algebras. They were introduced in [10], and we start giving a brief overview of their main properties.

Let k be a field. A k -algebra R is said to be an FCR-algebra provided every finite dimensional representation of R is completely reducible and the intersection of the kernels of all finite dimensional representations of R is zero. Basic examples of FCR-algebras are the universal enveloping algebras of finite-dimensional semisimple Lie algebras over a field of characteristic zero and their quantum deformations. For a list of examples see [11].

To explain better this setting we recall the following result from [11].

Theorem 7.1. (See [11, Theorem 3].) *Let R be a k -algebra where k is an arbitrary field. Then the following assertions are equivalent:*

- (i) *Every finite dimensional representation of R is completely reducible.*
- (ii) *Every two-sided ideal I of R of finite codimension is an idempotent ideal.*
- (iii) *If \mathcal{M}_1 and \mathcal{M}_2 are two-sided maximal ideals of finite codimension then*

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}_1 \cdot \mathcal{M}_2 = \mathcal{M}_2 \cdot \mathcal{M}_1.$$

It follows from Theorem 7.1, that every FCR-algebra R has a system of nonzero idempotent ideals \mathcal{S} such that for every $I \in \mathcal{S}$ the ring R/I is semisimple artinian and $\bigcap_{I \in \mathcal{S}} I = 0$.

It seems to be quite challenging to control countably generated projective modules over a (noetherian) FCR-algebra R . There are plenty of semisimple artinian factors of R where we can try to distinguish non-isomorphic projective R -modules. Unfortunately, these semisimple factors do not provide full information about projective modules over R . For example, if $R = U(\mathfrak{sl}_2(\mathbb{C}))$, then there exists a projective right ideal I which is stably free but not free (see [21]). Therefore, $I \otimes_R R/I_k$ is a free module for every $k \in \mathbb{N}$.

Since a noetherian FCR-algebra has plenty of idempotent ideals, it follows from Corollary 2.7 that these ideals are traces of countably generated projective modules. In the case R is the enveloping algebra of a semisimple Lie algebra, the only trace ideal of a nonzero finitely generated projective module is R . This is a result proved in Puninski's notes [17, Lemma 8.1] where it is attributed to Stafford. Since [17] has remained unpublished until now, we include a proof of this result.

Lemma 7.2. *Let R be the enveloping algebra of a finite dimensional Lie algebra over a field k . Then every non-zero finitely generated projective right R -module is a generator.*

In particular, the only idempotent ideals of R that are traces of finitely generated projective right modules are 0 and R .

Proof. By a result of Quillen [12], every non-zero finitely generated projective module P is stably free. Therefore, there exist n and m such that $P \oplus R^n \cong R^m$. Since R is noetherian, R_R has finite Goldie dimension, hence $n < m$.

Let I be the trace ideal of P . Since $P = PI$, the above isomorphism, implies that there is an isomorphism of right R -modules $(R/I)^n \cong (R/I)^m$. Since R_R is noetherian, $I = R$. \square

Theorem 7.3. *Let R be an indecomposable non-finite dimensional, noetherian, FCR-algebra over a field k . Let $I_0 \subseteq R$ be an ideal of finite codimension. Then there are uncountably many non-isomorphic countably generated projective R -modules that are not fair-sized and with trace ideal contained in I_0 .*

Proof. The hypothesis ensure that we can construct a strictly descending chain

$$I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_k \supsetneq \cdots$$

of idempotent ideals of R . Now the result follows as an application of Lemma 6.6. \square

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